Notes on Lie Sphere Geometry and the Cyclides of Dupin

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Abstract

In these notes, we give a detailed account of the method for studying Dupin hypersurfaces in $\mathbf{R}^n$ or S^n using Lie sphere geometry, and we conclude with a classification of the cyclides of Dupin obtained by using this approach.

Specifically, an oriented hypersurface $M^{n-1} \subset \mathbf{R}^n$ is a cyclide of Dupin of characteristic (p,q), where $p+q=n-1$, if M^{n-1} has two distinct principal curvatures at each point with respective multiplicities p and q, and each principal curvature function is constant along each leaf of its corresponding principal foliation. We show that every connected cyclide of characteristic (p,q) is contained in a unique compact, connected cyclide of characteristic (p,q). Furthermore, every compact, connected cyclide of characteristic (p,q) is equivalent by a Lie sphere transformation to a standard product of two spheres $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n$. As a corollary, we also derive a Möbius geometric classification of the cyclides in $\mathbf{R}^n$.

1 Introduction

In 1872, Lie [27] introduced his geometry of oriented hyperspheres in Euclidean space $\mathbf{R}^n$ in the context of his work on contact transformations (see [28]). Lie established a bijective correspondence between the set of all *Lie spheres*, i.e., oriented hyperspheres, oriented hyperplanes and point spheres, in $\mathbf{R}^n \cup \{\infty\}$, and the set of all points on the quadric hypersurface Q^{n+1} in real projective space P^{n+2} given by the equation $\langle x, x \rangle = 0$, where $\langle \, , \, \rangle$ is an indefinite scalar product with signature $(n+1, 2)$ on $\mathbf{R}^{n+3}$ given by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+2} y_{n+2} - x_{n+3} y_{n+3}, \tag{1}$$

for $x = (x_1, \ldots, x_{n+3})$, $y = (y_1, \ldots, y_{n+3})$.

Using linear algebra, one can show that this *Lie quadric* Q^{n+1} contains projective lines but no linear subspaces of P^{n+2} of higher dimension, since the metric in equation (1) has signature $(n+1, 2)$ (see [9, p. 21]). The one-parameter family of Lie spheres in $\mathbf{R}^n \cup \{\infty\}$ corresponding to the points on a line on Q^{n+1} is called a *parabolic pencil* of spheres. It consists of all Lie spheres in oriented contact at a certain contact element (p, N) on $\mathbf{R}^n \cup \{\infty\}$, where p is a point in $\mathbf{R}^n \cup \{\infty\}$ and N is a unit tangent vector to $\mathbf{R}^n \cup \{\infty\}$ at p. That is, (p, N) is an element of the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$. In this way, Lie also established a bijective correspondence between the manifold of contact elements on $\mathbf{R}^n \cup \{\infty\}$ and the manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1}.

A *Lie sphere transformation* is a projective transformation of P^{n+2} which maps the Lie quadric Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$, a Lie sphere transformation maps Lie spheres to Lie spheres. Furthermore, since a projective transformation maps lines to lines, a Lie sphere transformation preserves oriented contact of Lie spheres in $\mathbf{R}^n$.

Let $\mathbf{R}^{n+3}_2$ denote $\mathbf{R}^{n+3}$ endowed with the metric $\langle \, , \, \rangle$ in equation (1), and let $O(n+1, 2)$ denote the group of orthogonal transformations of $\mathbf{R}^{n+3}_2$. One can show that every Lie sphere transformation is the projective transformation induced by an orthogonal transformation, and thus the group G of Lie sphere transformations is isomorphic to the quotient group $O(n+1, 2)/\{\pm I\}$ (see [9, pp. 26–27]). Furthermore, any Möbius (conformal) transformation of $\mathbf{R}^n \cup \{\infty\}$ induces a Lie sphere transformation, and the Möbius group is precisely the subgroup of G consisting of all Lie sphere transformations that map point spheres to point spheres.

The manifold Λ^{2n-1} of projective lines on the quadric Q^{n+1} has a contact structure, i.e., a 1-form ω such that $\omega \wedge (d\omega)^{n-1}$ does not vanish on Λ^{2n-1}. The condition $\omega = 0$ defines a codimension one distribution D on Λ^{2n-1} which has integral submanifolds of dimension $n-1$, but none of higher dimension. Such an integral submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of D of maximal dimension is called a *Legendre submanifold*. If α is a Lie sphere transformation, then α maps lines on Q^{n+1} to lines on Q^{n+1}, and the map $\mu = \alpha\lambda$ is also a Legendre submanifold. The submanifolds λ and μ are said to be *Lie equivalent*.

Let M^{n-1} be an oriented hypersurface in $\mathbf{R}^n$. Then M^{n-1} naturally induces a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$, called the *Legendre lift of* M^{n-1}, as we will show in these notes. More generally, an immersed submanifold V of codimension greater than one in $\mathbf{R}^n$ induces a Legendre lift whose domain is the unit normal bundle B^{n-1} of V in $\mathbf{R}^n$. Similarly, any submanifold of the unit sphere $S^n \subset \mathbf{R}^{n+1}$ has a Legendre lift. We can relate properties of a submanifold of $\mathbf{R}^n$ or S^n to Lie geometric properties of its Legendre lift, and attempt to classify certain types of Legendre submanifolds up to Lie sphere transformations. This, in turn, gives classification results for the corresponding classes of Euclidean submanifolds of $\mathbf{R}^n$ or S^n.

We next recall some basic ideas from Euclidean submanifold theory. For an oriented hypersurface $f : M \to \mathbf{R}^n$ with field of unit normal vectors ξ, the eigenvalues of the shape operator (second fundamental form) A of M are called *principal curvatures*, and their corresponding eigenspaces are called *principal spaces*. A submanifold S of M is called a *curvature surface* of M if at each point x of S, the tangent space T_xS is a principal space at x. This generalizes the classical notion of a line of curvature of a surface in $\mathbf{R}^3$. If κ is a non-zero principal curvature of M at x, the point

$$f_\kappa(x) = f(x) + (1/\kappa)\xi(x) \tag{2}$$

is called the *focal point* of M at x determined by κ. The hypersphere in $\mathbf{R}^n$ tangent to M at $f(x)$ and centered at the focal point $f_\kappa(x)$ is called the *curvature sphere* at x determined by κ.

It is well known that there always exists an open dense subset Ω of M on which the multiplicities of the principal curvatures are locally constant (see, for example, Singley [42]). If a principal curvature κ has constant multiplicity m in some open set $U \subset M$, then the corresponding m-dimensional distribution of principal spaces is integrable, i.e., it is an m-dimensional foliation, and the leaves of this *principal foliation* are curvature surfaces. Furthermore, if the multiplicity m of κ is greater than one, then by using the Codazzi equation, one can show that κ is constant along each leaf of this principal foliation (see, for example, [19, p. 24]). This is not true, in general, if the multiplicity $m = 1$.

A hypersurface M in $\mathbf{R}^n$ or S^n is said to be *Dupin* if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface is said to be *proper Dupin* if each principal curvature has constant multiplicity on M, i.e., the number of distinct principal curvatures is constant on M (see Pinkall [40]).

A well known class of proper Dupin hypersurfaces are the cylides of Dupin in $\mathbf{R}^3$, introduced by Dupin [21] in 1822. Dupin defined a cyclide to be a surface M in $\mathbf{R}^3$ that is the envelope of the family of spheres tangent to three fixed spheres in $\mathbf{R}^3$. This is equivalent to requiring that M have two distinct principal curvatures at each point, and that both focal maps of M degenerate into curves (instead of surfaces). Then M is the envelope of the family of curvature spheres centered along each of the focal curves. The three fixed spheres in Dupin's definition can be chosen to be three spheres from either family of curvature spheres.

The most basic examples of cyclides of Dupin in $\mathbf{R}^3$ are a torus of revolution, a circular cylinder, and a circular cone. The proper Dupin property is easily shown to be invariant under Möbius transformations of $\mathbf{R}^3 \cup \{\infty\}$, and it turns out that all cyclides of Dupin in $\mathbf{R}^3$ can be obtained from these three types of examples by inversion in a sphere in $\mathbf{R}^3$ (see, for example, [18, pp. 151–166]).

Using standard techniques of surface theory in $\mathbf{R}^3$, one can show that the two focal curves of a cyclide of Dupin are a pair of *focal conics*. These are, by definition, either an ellipse and hyperbola in mutually orthogonal planes such that the vertices of the ellipse are the foci of the hyperbola and vice-versa, or a pair of parabolas in orthogonal planes such that the vertex of each is the focus of the other. Also possible is the degenerate case consisting of a circle and a straight line (covered twice) through the center of the circle and orthogonal to the plane of the circle, as with the focal curves of a torus of revolution.

Another classification states that every connected cyclide in $\mathbf{R}^3$ is Möbius equivalent to an open subset of a surface of revolution obtained by revolving a profile circle $S^1 \subset \mathbf{R}^2$ about an axis $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \mathbf{R}^3$. The profile circle is allowed to intersect the axis, thereby introducing Euclidean singularities. However, the corresponding Legendre map into the space of contact elements in $\mathbf{R}^3 \cup \{\infty\}$ is an immersion. We will prove this result using the techniques of Lie sphere geometry in Theorem 10.2, and then give a complete Möbius geometric description of the cyclides in Section 10.

The classical cyclides of Dupin in $\mathbf{R}^3$ were studied intensively by many leading mathematicians in the nineteenth century, including Liouville [30], Cayley [6], and Maxwell [31], whose paper contains stereoscopic figures of the various types of cyclides. A good account of the history of the cyclides in the nineteenth century is given by Lilienthal [29] (see also Klein [26, pp. 56–58], Darboux [20, vol. 2, pp. 267–269], Blaschke [4, p. 238], Eisenhart

[22, pp. 312–314], Hilbert and Cohn-Vossen [25, pp. 217–219], Fladt and Baur [23, pp. 354–379], and Cecil and Ryan [18, pp. 151–166]).

The notions of Dupin and proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n can be generalized to a class of Legendre submanifolds in Lie sphere geometry known as Dupin submanifolds. In particular, the Legendre lifts of Dupin hypersurfaces in $\mathbf{R}^n$ or S^n are Dupin submanifolds in this generalized sense. The Dupin and proper Dupin properties of such submanifolds are easily seen to be invariant under Lie sphere transformations. This makes Lie sphere geometry a particularly effective setting for the study of Dupin hypersurfaces, and many classifications of Dupin hypersurfaces have been obtained up to Lie sphere transformations.

In particular, Pinkall's paper [40] describing the higher dimensional cyclides of Dupin in the context of Lie sphere geometry was particularly influential, and it had its roots in Volume 3 of the book of Blaschke [4], which studied surfaces in the context of Lie sphere geometry. For any two positive integers p and q with $p + q = n - 1$, Pinkall defined a *cyclide of Dupin of characteristic* (p, q) to be a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with two distinct curvature spheres of respective multiplicities p and q at each point. In these notes, we present Pinkall's [40] classification of the cyclides of Dupin of arbitrary dimension in $\mathbf{R}^n$ or S^n (Theorem 10.1), obtained by using the methods of Lie sphere geometry.

Specifically, we show that any connected cyclide of Dupin of characteristic (p, q) is contained in a unique compact, connected cyclide of Dupin of characteristic (p, q). Furthermore, every compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}. \tag{3}$$

As a corollary, we also derive a Möbius geometric classification (Theorem 10.2) of the higher dimensional cyclides in $\mathbf{R}^n$, similar to the Möbius geometric classification of the cyclides in $\mathbf{R}^3$ mentioned above.

Of course, a standard product of two spheres in S^n is an isoparametric hypersurface in S^n, i.e., it has constant principal curvatures in S^n. The images of isoparametric hypersurfaces in S^n under stereographic projection from S^n to $\mathbf{R}^n$ form a particularly important class of proper Dupin hypersurfaces in $\mathbf{R}^n$. Many important results in the field deal with relationships between compact proper Dupin hypersurfaces and isoparametric hypersurfaces in spheres, including the question of which compact proper Dupin hypersurfaces are Lie equivalent to an isoparametric hypersurface in a sphere (see Theorem 9.3 for necessary and sufficient conditions for this to be the case).

For more detail on compact proper Dupin hypersurfaces and their relationship to isoparametric hypersurfaces, see the papers of Pinkall [38]–[40], Pinkall and Thorbergsson [41], Thorbergsson [44], Miyaoka [32]–[34], Miyaoka and Ozawa [35], Banchoff [2], Grove and Halperin [24], Stolz [43], Niebergall [36]–[37], Cecil and Chern [11]–[12], Cecil, Chi and Jensen [13], Cecil and Jensen [14]–[15], and Cecil and Ryan [16]–[17], among others. See also the surveys of Thorbergsson [45], Cecil [10], Cecil and Ryan [19, pp. 308–322], and the introduction of the book [9, pp. 1–7]).

These notes are based primarily on the author's book [9], and several passages in these notes are taken directly from that book.

2 Notation and Preliminary Results

Let (x, y) be the indefinite scalar product on the Lorentz space $\mathbf{R}^{n+1}_1$ defined by

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}, \tag{4}$$

where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We will call this scalar product the *Lorentz metric*. A vector x is said to be *spacelike*, *timelike* or *lightlike*, respectively, depending on whether (x, x) is positive, negative or zero. We will use this terminology even when we are using an indefinite metric of different signature.

In Lorentz space, the set of all lightlike vectors, given by the equation,

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2, \tag{5}$$

forms a cone of revolution, called the *light cone*. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone."

If x is a nonzero vector in $\mathbf{R}^{n+1}_1$, let $x^\perp$ denote the orthogonal complement of x with respect to the Lorentz metric. If x is timelike, then the metric restricts to a positive definite form on $x^\perp$, and $x^\perp$ intersects the light cone only at the origin. If x is spacelike, then the metric has signature $(n-1, 1)$ on $x^\perp$, and $x^\perp$ intersects the cone in a cone of one less dimension. If x is lightlike, then $x^\perp$ is tangent to the cone along the line through the origin determined by x. The metric has signature $(n-1, 0)$ on this n-dimensional plane.

Lie sphere geometry is defined in the context of real projective space P^n, so we now briefly review some important concepts from projective geometry. We define an equivalence relation on $\mathbf{R}^{n+1} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t. We denote the equivalence class determined by a vector x by $[x]$. Projective space P^n is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in $\mathbf{R}^{n+1}$. The rectangular coordinates $(x_1, \ldots, x_{n+1})$ are called *homogeneous coordinates* of the point $[x]$, and they are only determined up to a nonzero scalar multiple. The affine space $\mathbf{R}^n$ can be embedded in P^n as the complement of the hyperplane $(x_1 = 0)$ at infinity by the map $\phi : \mathbf{R}^n \to P^n$ given by $\phi(u) = [(1, u)]$. A scalar product on $\mathbf{R}^{n+1}$, such as the Lorentz metric, determines a polar relationship between points and hyperplanes in P^n. We will also use the notation $x^\perp$ to denote the polar hyperplane of $[x]$ in P^n, and we will call $[x]$ the *pole* of $x^\perp$.

If x is a non-zero lightlike vector in $\mathbf{R}^{n+1}_1$, then $[x]$ can be represented by a vector of the form $(1, u)$ for $u \in \mathbf{R}^n$. Then the equation $(x, x) = 0$ for the light cone becomes $u \cdot u = 1$ (Euclidean dot product), i.e., the equation for the unit sphere in $\mathbf{R}^n$. Hence, the set of points in P^n determined by lightlike vectors in $\mathbf{R}^{n+1}_1$ is naturally diffeomorphic to the sphere S^{n-1}.

3 Möbius Geometry of Unoriented Spheres

As a first step toward Lie's geometry of oriented spheres, we recall the geometry of unoriented spheres in $\mathbf{R}^n$ known as "Möbius" or "conformal" geometry. We will always assume that $n \geq 2$. In this section, we will only consider spheres and planes of codimension one, and we will often omit the prefix "hyper," from the words "hypersphere" and "hyperplane."

We denote the Euclidean dot product of two vectors u and v in $\mathbf{R}^n$ by $u \cdot v$. We first consider stereographic projection $\sigma : \mathbf{R}^n \to S^n - \{P\}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$ given by $y \cdot y = 1$, and $P = (-1, 0, \ldots, 0)$ is the south pole of S^n. The well-known formula for $\sigma(u)$ is

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right).$$

Note that σ is sometimes referred to as "inverse stereographic projection," in which case its inverse map from $S^n - \{P\}$ to $\mathbf{R}^n$ is called "stereographic projection."

We next embed $\mathbf{R}^{n+1}$ into P^{n+1} by the embedding ϕ mentioned in the previous section. Thus, we have the map $\phi\sigma : \mathbf{R}^n \to P^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1-u\cdot u}{1+u\cdot u}, \frac{2u}{1+u\cdot u}\right)\right] = \left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u\right)\right]. \qquad (6)$$

Let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on P^{n+1} and $(\ ,\ )$ the Lorentz metric on the space $\mathbf{R}_1^{n+2}$. Then $\phi\sigma(\mathbf{R}^n)$ is just the set of points in P^{n+1} lying on the n-sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \ldots, 0)]$ corresponding to the south pole P. We will refer to the points in Σ other than $[(1, -1, 0, \ldots, 0)]$ as *proper points*, and will call Σ the *Möbius sphere* or *Möbius space*. At times, it is easier to simply begin with S^n rather than $\mathbf{R}^n$ and thus avoid the need for the map σ and the special point P. However, there are also advantages for beginning in $\mathbf{R}^n$.

The basic framework for the Möbius geometry of unoriented spheres is as follows. Suppose that ξ is a spacelike vector in $\mathbf{R}_1^{n+2}$. Then the polar hyperplane $\xi^\perp$ to $[\xi]$ in P^{n+1} intersects the sphere Σ in an $(n-1)$-sphere S^{n-1}. The sphere S^{n-1} is the image under $\phi\sigma$ of an $(n-1)$-sphere in $\mathbf{R}^n$, unless it contains the improper point $[(1, -1, 0, \ldots, 0)]$, in which case it is the image under $\phi\sigma$ of a hyperplane in $\mathbf{R}^n$. Hence, we have a bijective correspondence between the set of all spacelike points in P^{n+1} and the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$.

It is often useful to have specific formulas for this correspondence. Consider the sphere in $\mathbf{R}^n$ with center p and radius $r > 0$ given by the equation

$$(u-p)\cdot(u-p) = r^2. \qquad (7)$$

We wish to translate this into an equation involving the Lorentz metric and the corresponding polarity relationship on P^{n+1}. A direct calculation shows that equation (7) is equivalent to the equation

$$(\xi, \phi\sigma(u)) = 0, \qquad (8)$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p\right), \qquad (9)$$

and $\phi\sigma(u)$ is given by equation (6). Thus, the point u is on the sphere given by equation (7) if and only if $\phi\sigma(u)$ lies on the polar hyperplane of

$[\xi]$. Note that the first two coordinates of ξ satisfy $\xi_1 + \xi_2 = 1$, and that $(\xi, \xi) = r^2$. Although ξ is only determined up to a nonzero scalar multiple, we can conclude that $\eta_1 + \eta_2$ is not zero for any $\eta \simeq \xi$.

Conversely, given a spacelike point $[z]$ with $z_1 + z_2$ nonzero, we can determine the corresponding sphere in $\mathbf{R}^n$ as follows. Let $\xi = z/(z_1 + z_2)$ so that $\xi_1 + \xi_2 = 1$. Then from equation (9), the center of the corresponding sphere is the point $p = (\xi_3, \ldots, \xi_{n+2})$, and the radius is the square root of (ξ, ξ).

Next suppose that η is a spacelike vector with $\eta_1 + \eta_2 = 0$. Then

$$(\eta, (1, -1, 0, \ldots, 0)) = 0.$$

Thus, the improper point $\phi(P)$ lies on the polar hyperplane of $[\eta]$, and the point $[\eta]$ corresponds to a hyperplane in $\mathbf{R}^n$. Again we can find an explicit correspondence. Consider the hyperplane in $\mathbf{R}^n$ given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{10}$$

A direct calculation shows that (10) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \text{where } \eta = (h, -h, N). \tag{11}$$

Thus, the hyperplane (10) is represented in the polarity relationship by $[\eta]$.

Conversely, let z be a spacelike point with $z_1 + z_2 = 0$. Then $(z, z) = v \cdot v$, where $v = (z_3, \ldots, z_{n+2})$. Let $\eta = z/|v|$. Then η has the form (11) and $[z]$ corresponds to the hyperplane (10). Thus we have explicit formulas for the bijective correspondence between the set of spacelike points in P^{n+1} and the set of hyperspheres and hyperplanes in $\mathbf{R}^n$.

Similarly, we can construct a bijective correspondence between the space of all hyperspheres in the unit sphere $S^n \subset \mathbf{R}^{n+1}$ and the manifold of all spacelike points in P^{n+1} as follows. The hypersphere S in S^n with center $p \in S^n$ and (spherical) radius $\rho, 0 < \rho < \pi$, is given by the equation

$$p \cdot y = \cos \rho, \quad 0 < \rho < \pi, \tag{12}$$

for $y \in S^n$. If we take $[z] = \phi(y) = [(1, y)]$, then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)},$$

where $e_1 = (1, 0, \ldots, 0)$. Thus equation (12) is equivalent to the equation

$$(z, (\cos \rho, p)) = 0, \tag{13}$$

in homogeneous coordinates in P^{n+1}. Therefore, y lies on the hypersphere S given by equation (12) if and only if $[z] = \phi(y)$ lies on the polar hyperplane in P^{n+1} of the spacelike point

$$[\xi] = [(\cos\rho, p)]. \tag{14}$$

Remark 3.1. *In these notes, we will focus on spheres in* $\mathbf{R}^n$ *or* S^n*. See [9, pp. 16–18] for a treatment of the geometry of hyperspheres in real hyperbolic space* H^n*.*

Of course, the fundamental invariant of Möbius geometry is the angle. The study of angles in this setting is quite natural, since orthogonality between spheres and planes in $\mathbf{R}^n$ can be expressed in terms of the Lorentz metric. Let S_1 and S_2 denote the spheres in $\mathbf{R}^n$ with respective centers p_1 and p_2 and respective radii r_1 and r_2. By the Pythagorean Theorem, the two spheres intersect orthogonally if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{15}$$

If these spheres correspond by equation (9) to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, then a calculation shows that equation (15) is equivalent to the condition

$$(\xi_1, \xi_2) = 0. \tag{16}$$

A hyperplane π in $\mathbf{R}^n$ is orthogonal to a hypersphere S precisely when π passes through the center of S. If S has center p and radius r, and π is given by the equation $u \cdot N = h$, then the condition for orthogonality is just $p \cdot N = h$. If S corresponds to $[\xi]$ as in (9) and π corresponds to $[\eta]$ as in (11), then this equation for orthogonality is equivalent to $(\xi, \eta) = 0$. Finally, if two planes π_1 and π_2 are represented by $[\eta_1]$ and $[\eta_2]$ as in (11), then the orthogonality condition $N_1 \cdot N_2 = 0$ is equivalent to the equation $(\eta_1, \eta_2) = 0$. Thus, in all cases of hyperspheres or hyperplanes in $\mathbf{R}^n$, orthogonal intersection corresponds to a polar relationship in P^{n+1} given by equations (8) or (11).

We conclude this section with a discussion of Möbius transformations. Recall that a linear transformation $A \in GL(n+2)$ induces a projective transformation $P(A)$ on P^{n+1} defined by $P(A)[x] = [Ax]$. The map P is a homomorphism of $GL(n+2)$ onto the group $PGL(n+1)$ of projective transformations of P^{n+1}, and its kernel is the group of nonzero multiples of the identity transformation $I \in GL(n+2)$.

A *Möbius transformation* is a projective transformation α of P^{n+1} that preserves the condition $(\eta, \eta) = 0$ for $[\eta] \in P^{n+1}$, that is, $\alpha = P(A)$, where $A \in GL(n+2)$ maps lightlike vectors in $\mathbf{R}^{n+2}_1$ to lightlike vectors. It can be shown (see, for example, [9, pp. 26–27]) that such a linear transformation A is a nonzero scalar multiple of a linear transformation $B \in O(n+1,1)$, the orthogonal group for the Lorentz inner product space $\mathbf{R}^{n+2}_1$. Thus, $\alpha = P(A) = P(B)$.

The Möbius transformation $\alpha = P(B)$ induced by an orthogonal transformation $B \in O(n+1,1)$ maps spacelike points to spacelike points in P^{n+1}, and it preserves the polarity condition $(\xi, \eta) = 0$ for any two points $[\xi]$ and $[\eta]$ in P^{n+1}. Therefore by the correspondence given in equations (8) and (11) above, α maps the set of hyperspheres and hyperplanes in $\mathbf{R}^n$ to itself, and it preserves orthogonality and hence angles between hyperspheres and hyperplanes. A similar statement holds for the set of all hyperspheres in S^n.

Let H denote the group of Möbius transformations and let

$$\psi : O(n+1,1) \to H \tag{17}$$

be the restriction of the map P to $O(n+1,1)$. The discussion above shows that ψ is onto, and the kernel of ψ is $\{\pm I\}$, the intersection of $O(n+1,1)$ with the kernel of P. Therefore, H is isomorphic to the quotient group $O(n+1,1)/\{\pm I\}$.

One can show that the group H is generated by Möbius transformations induced by inversions in spheres in $\mathbf{R}^n$. This follows from the fact that the corresponding orthogonal groups are generated by reflections in hyperplanes. In fact, every orthogonal transformation on an indefinite inner product space $\mathbf{R}^n_k$ is a product of at most n reflections, a result due to Cartan and Dieudonné. (See Cartan [5, pp. 10–12], Chapter 3 of Artin's book [1], or [9, pp. 30–34]).

Since a Möbius transformation $\alpha = P(B)$ for $B \in O(n+1,1)$ maps lightlike points to lightlike points in P^{n+1} in a bijective way, it induces a diffeomorphism of the n-sphere Σ which is conformal by the considerations given above. It is well known that the group of conformal diffeomorphisms of the n-sphere is precisely the Möbius group.

4 Lie Geometry of Oriented Spheres

We now turn to the construction of Lie's geometry of oriented spheres in $\mathbf{R}^n$. Let W^{n+1} be the set of vectors in $\mathbf{R}^{n+2}_1$ satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in $\mathbf{R}^{n+2}_1$. If α is a spacelike point in P^{n+1}, then there are precisely two vectors $\pm\zeta$ in W^{n+1} with $\alpha = [\zeta]$. These two vectors can be taken to correspond to the two orientations of the oriented sphere or plane represented by α, as we will now describe. We first introduce one more coordinate. We embed $\mathbf{R}^{n+2}_1$ into P^{n+2} by the embedding $z \mapsto [(z, 1)]$. If $\zeta \in W^{n+1}$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta, 1)]$ in P^{n+2} lies on the quadric Q^{n+1} in P^{n+2} given in homogeneous coordinates by the equation

$$\langle x, x \rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0. \tag{18}$$

The manifold Q^{n+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (18) is called the *Lie metric* or *Lie scalar product*. We will let $\{e_1, \ldots, e_{n+3}\}$ denote the standard orthonormal basis for the scalar product space $\mathbf{R}^{n+3}_2$ with metric $\langle\ ,\ \rangle$. Here e_1 and e_{n+3} are timelike and the rest are spacelike.

We shall now see how points on Q^{n+1} correspond to the set of oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$. Suppose that x is any point on the quadric with homogeneous coordinate $x_{n+3} \neq 0$. Then x can be represented by a vector of the form $(\zeta, 1)$, where the Lorentz scalar product $(\zeta, \zeta) = 1$. Suppose first that $\zeta_1 + \zeta_2 \neq 0$. Then in Möbius geometry $[\zeta]$ represents a sphere in $\mathbf{R}^n$. If as in equation (9), we represent $[\zeta]$ by a vector of the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right),$$

then $(\xi, \xi) = r^2$. Thus ζ must be one of the vectors $\pm\xi/r$. In P^{n+2}, we have

$$[(\zeta, 1)] = [(\pm\xi/r, 1)] = [(\xi, \pm r)].$$

We can interpret the last coordinate as a signed radius of the sphere with center p and unsigned radius $r > 0$. In order to interpret this geometrically, we adopt the convention that a positive signed radius corresponds to the orientation of the sphere represented by the inward field of unit normals, and a negative signed radius corresponds to the orientation given by the outward field of unit normals. Hence, the two orientations of the sphere in $\mathbf{R}^n$ with center p and unsigned radius $r > 0$ are represented by the two projective points,

$$\left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r\right)\right] \tag{19}$$

in Q^{n+1}. Next if $\zeta_1 + \zeta_2 = 0$, then $[\zeta]$ represents a hyperplane in $\mathbf{R}^n$, as in equation (11). For $\zeta = (h, -h, N)$, with $|N| = 1$, we have $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \tag{20}$$

These represent the two orientations of the plane with equation $u \cdot N = h$. We make the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N, while the orientation given by $-N$ corresponds to the point $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$.

Thus far we have determined a bijective correspondence between the set of points x in Q^{n+1} with $x_{n+3} \neq 0$ and the set of all oriented spheres and planes in $\mathbf{R}^n$. Suppose now that $x_{n+3} = 0$, i.e., consider a point $[(z, 0)]$, for $z \in \mathbf{R}^{n+2}_1$. Then $\langle x, x\rangle = (z, z) = 0$, and $[z] \in P^{n+1}$ is simply a point of the Möbius sphere Σ. Thus we have the following bijective correspondence between objects in Euclidean space and points on the Lie quadric:

Euclidean	**Lie**	
points: $u \in \mathbf{R}^n$	$\left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u, 0\right)\right]$	
∞	$[(1, -1, 0, 0)]$	(21)
spheres: center p, signed radius r	$\left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, r\right)\right]$	
planes: $u \cdot N = h$, unit normal N	$[(h, -h, N, 1)]$	

In Lie sphere geometry, points are considered to be spheres of radius zero, or "point spheres."

From now on, we will use the term *Lie sphere* or simply "sphere" to denote an oriented sphere, oriented plane or a point sphere in $\mathbf{R}^n \cup \{\infty\}$. We will refer to the coordinates on the right side of equation (21) as the *Lie coordinates* of the corresponding point, sphere or plane. In the case of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively, these coordinates were classically called *pentaspherical* and *hexaspherical* coordinates (see Blaschke [4]).

At times it is useful to have formulas to convert Lie coordinates back into Cartesian equations for the corresponding Euclidean object. Suppose first that $[x]$ is a point on the Lie quadric with $x_1 + x_2 \neq 0$. Then $x = \rho y$, for some $\rho \neq 0$, where y is one of the standard forms on the right side of the table above. From the table, we see that $y_1 + y_2 = 1$, for all proper points and all spheres. Hence if we divide x by $x_1 + x_2$, the new vector will be in standard form, and we can read off the corresponding Euclidean object from the table. In particular, if $x_{n+3} = 0$, then $[x]$ represents the point sphere $u = (u_3, \ldots, u_{n+2})$ where

$$u_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2. \tag{22}$$

If $x_{n+3} \neq 0$, then $[x]$ represents the sphere with center $p = (p_3, \ldots, p_{n+2})$ and signed radius r given by

$$p_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2; \quad r = x_{n+3}/(x_1 + x_2). \tag{23}$$

Finally, suppose that $x_1 + x_2 = 0$. If $x_{n+3} = 0$, then the equation $\langle x, x \rangle = 0$ forces x_i to be zero for $3 \leq i \leq n+2$. Thus $[x] = [(1, -1, 0, \ldots, 0)]$, the improper point. If $x_{n+3} \neq 0$, we divide x by x_{n+3} to make the last coordinate 1. Then if we set $N = (N_3, \ldots, N_{n+2})$ and h according to

$$N_i = x_i/x_{n+3}, \quad 3 \leq i \leq n+2; \quad h = x_1/x_{n+3}, \tag{24}$$

the conditions $\langle x, x \rangle = 0$ and $x_1 + x_2 = 0$ force N to have unit length. Thus $[x]$ corresponds to the hyperplane $u \cdot N = h$, with unit normal N and h as in equation (24).

If we wish to consider oriented hyperspheres and point spheres in the unit sphere S^n in $\mathbf{R}^{n+1}$, then the table above can be simplified. First, we have shown that in Möbius geometry, the unoriented hypersphere S in S^n with center $p \in S^n$ and spherical radius ρ, $0 < \rho < \pi$, corresponds to the point $[\xi] = [(\cos\rho, p)]$ in P^{n+1}. To correspond the two orientations of this sphere to points on the Lie quadric, we first note that

$$(\xi, \xi) = -\cos^2\rho + 1 = \sin^2\rho.$$

Since $\sin\rho > 0$ for $0 < \rho < \pi$, we can divide ξ by $\sin\rho$ and consider the two vectors $\zeta = \pm\xi/\sin\rho$ that satisfy $(\zeta, \zeta) = 1$. We then map these two points into the Lie quadric to get the points

$$[(\zeta, 1)] = [(\xi, \pm\sin\rho)] = [(\cos\rho, p, \pm\sin\rho)].$$

in Q^{n+1}. We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0$, where $-\pi < \rho < \pi$, and center p corresponds to the point

$$[x] = [(\cos\rho, p, \sin\rho)]. \tag{25}$$

in Q^{n+1}. This formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$.

We adopt the convention that the positive radius ρ in (25) corresponds to the orientation of the sphere given by the field of unit normals which are tangent vectors to geodesics in S^n from $-p$ to p, and a negative radius corresponds to the opposite orientation. Each oriented sphere can be considered in two ways, with center p and signed radius $\rho, -\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

For a given point $[x]$ in the quadric Q^{n+1}, we can determine the corresponding oriented hypersphere or point sphere in S^n as follows. Multiplying by -1, if necessary, we can arrange that that the first coordinate x_1 of x is nonnegative. If x_1 is positive, then it follows from equation (25) that the center p and signed radius $\rho, -\pi/2 < \rho < \pi/2$, are given by

$$\tan\rho = x_{n+3}/x_1, \quad p = (x_2, \ldots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \tag{26}$$

If $x_1 = 0$, then x_{n+3} is nonzero, and we can divide by x_{n+3} to obtain a point with coordinates $(0, p, 1)$. This corresponds to the oriented hypersphere in S^n with center p and signed radius $\pi/2$, which is a great sphere in S^n.

Remark 4.1. *In a similar way, one can develop the Lie sphere geometry of oriented spheres in real hyperbolic space H^n (see, for example, [9, p. 18]).*

5 Oriented Contact of Spheres

As we saw in Section 3, the angle between two spheres is the fundamental geometric quantity in Möbius geometry, and it is the quantity that is preserved by Möbius transformations. In Lie's geometry of oriented spheres, the corresponding fundamental notion is that of oriented contact of spheres. By definition, two oriented spheres S_1 and S_2 in $\mathbf{R}^n$ are in *oriented contact* if they are tangent to each other, and they have the same orientation at the point of contact. There are two geometric possibilities depending on whether the signed radii of S_1 and S_2 have the same sign or opposite signs.

In either case, if p_1 and p_2 are the respective centers of S_1 and S_2, and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \tag{27}$$

Similarly, we say that an oriented hypersphere sphere S with center p and signed radius r and an oriented hyperplane π with unit normal N and equation $u \cdot N = h$ are in oriented contact if π is tangent to S and their orientations agree at the point of contact. This condition is given by the equation

$$p \cdot N = r + h. \tag{28}$$

Next we say that two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. These planes can be considered to be two oriented spheres in oriented contact at the improper point. Finally, a proper point u in $\mathbf{R}^n$ is in oriented contact with a sphere or a plane if it lies on the sphere or plane, and the improper point is in oriented contact with each plane, since it lies on each plane.

An important fact in Lie sphere geometry is that if S_1 and S_2 are two Lie spheres which are represented as in equation (21) by $[k_1]$ and $[k_2]$, then the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \tag{29}$$

This can be checked easily by a direct calculation.

By standard linear algebra in indefinite inner product spaces (see, for example, [9, p. 21]), the fact that the signature of $\mathbf{R}^{n+3}_2$ is $(n+1, 2)$ implies that the Lie quadric contains projective lines in P^{n+2}, but no linear subspaces of P^{n+2} of higher dimension. These projective lines on Q^{n+1} play a crucial role in the theory of submanifolds in the context of Lie sphere geometry.

One can show further (see [9, pp. 21–23]), that if $[k_1]$ and $[k_2]$ are two points of Q^{n+1}, then the line $[k_1, k_2]$ in P^{n+2} lies on Q^{n+1} if and only if the the spheres corresponding to $[k_1]$ and $[k_2]$ are in oriented contact, i.e., $\langle k_1, k_2 \rangle = 0$. Moreover, if the line $[k_1, k_2]$ lies on Q^{n+1}, then the set of spheres in $\mathbf{R}^n$ corresponding to points on the line $[k_1, k_2]$ is precisely the set of all spheres in oriented contact with both of these spheres. Such a 1-parameter family of spheres is called a *parabolic pencil* of spheres in $\mathbf{R}^n \cup \{\infty\}$.

Each parabolic pencil contains exactly one point sphere, and if that point sphere is a proper point, then the parabolic pencil contains exactly one hyperplane π in $\mathbf{R}^n$, and the pencil consists of all spheres in oriented contact with the oriented plane π at p. Thus, we can associate the parabolic pencil with the point (p, N) in the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$, where N is the unit normal to the oriented plane π.

If the point sphere in the pencil is the improper point, then the parabolic pencil is a family of parallel hyperplanes in oriented contact at the improper point. If N is the common unit normal to all of these planes, then we can associate the pencil with the point (∞, N) in the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$.

Similarly, we can establish a correspondence between parabolic pencils and elements of the unit tangent bundle $T_1 S^n$ that is expressed in terms of the spherical metric on S^n. If ℓ is a line on the quadric, then ℓ intersects both $e_1^\perp$ and $e_{n+3}^\perp$ at exactly one point, where $e_1 = (1, 0, \ldots, 0)$ and $e_{n+3} = (0, \ldots, 0, 1)$. So the parabolic pencil corresponding to ℓ contains exactly one point sphere (orthogonal to e_{n+3}) and one great sphere (orthogonal to e_1), given respectively by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)]. \tag{30}$$

Since ℓ lies on the quadric, we know that $\langle k_1, k_2 \rangle = 0$, and this condition is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p. Thus, the parabolic pencil of spheres corresponding to the line ℓ can be associated with the point (p, ξ) in $T_1 S^n$. More specifically, the line ℓ can be parametrized as

$$[K_t] = [\cos t \ k_1 + \sin t \ k_2] = [(\cos t, \cos t \ p + \sin t \ \xi, \sin t)]. \tag{31}$$

From equation (25) above, we see that $[K_t]$ corresponds to the oriented sphere in S^n with center

$$p_t = \cos t \ p + \sin t \ \xi, \tag{32}$$

and signed radius t. The pencil consists of all oriented spheres in S^n in oriented contact with the great sphere corresponding to $[k_2]$ at the point (p, ξ) in $T_1 S^n$. Their centers p_t lie along the geodesic in S^n with initial point p and initial velocity vector ξ. Detailed proofs of all these facts are given in [9, pp. 21–23].

We conclude this section with a discussion of Lie sphere transformations. By definition, a *Lie sphere transformation* is a projective transformation of P^{n+2} which maps the Lie quadric Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$ or S^n, a Lie sphere transformation maps Lie spheres to Lie spheres, and since it is a projective transformation, it maps lines on Q^{n+1} to lines on Q^{n+1}. Thus, it preserves oriented contact of spheres in $\mathbf{R}^n$ or S^n. Conversely, Pinkall [40] (see also [9, pp. 28–30]) proved the so-called "Fundamental Theorem of Lie sphere geometry," which states that any line preserving diffeomorphism of Q^{n+1} is the restriction to Q^{n+1} of a projective transformation, that is, a transformation of the space of oriented spheres which preserves oriented contact is a Lie sphere transformation.

By the same type of reasoning given in Section 3 for Möbius transformations, one can show that the group G of Lie sphere transformations is isomorphic to the group $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations of $\mathbf{R}^{n+3}_2$. As with the Möbius group, it follows from the theorem of Cartan and Dieudonné (see [9, pp. 30–34]) that the Lie sphere group G is generated by Lie inversions, that is, projective transformations that are induced by reflections in $O(n+1,2)$.

The Möbius group H can be considered to be a subgroup of G in the following manner. Each Möbius transformation on the space of unoriented spheres, naturally induces two Lie sphere transformations on the space Q^{n+1} of oriented spheres as follows. If A is in $O(n+1,1)$, then we can extend A to a transformation B in $O(n+1,2)$ by setting $B = A$ on $\mathbf{R}^{n+2}_1$ and $B(e_{n+3}) = e_{n+3}$. In terms the standard orthonormal basis in $\mathbf{R}^{n+3}_2$, the transformation B has the matrix representation,

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. \tag{33}$$

Although A and $-A$ induce the same Möbius transformation in H, the Lie transformation $P(B)$ is not the same as the Lie transformation $P(C)$ induced by the matrix

$$C = \begin{bmatrix} -A & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} A & 0 \\ 0 & -1 \end{bmatrix},$$

where $\simeq$ denotes equivalence as projective transformations. Note that $P(B) = \Gamma P(C)$, where Γ is the Lie transformation represented in matrix form by

$$\Gamma = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \simeq \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

From equation (21), we see that Γ has the effect of changing the orientation of every oriented sphere or plane. The transformation Γ is called the *change of orientation transformation* or "Richtungswechsel" in German. Hence, the two Lie sphere transformations induced by the Möbius transformation $P(A)$ differ by this change of orientation factor.

Thus, the group of Lie sphere transformations induced from Möbius transformations is isomorphic to $O(n+1,1)$. This group consists of those Lie transformations that map $[e_{n+3}]$ to itself, and it is a double covering of the Möbius group H. Since these transformations are induced from orthogonal transformations of $\mathbf{R}^{n+3}_2$, they also map $e^{\perp}_{n+3}$ to itself, and thereby map point spheres to point spheres. When working in the context of Lie sphere geometry, we will refer to these transformations as "Möbius transformations."

6 Legendre Submanifolds

The goal of this section is to define a contact structure on the unit tangent bundle T_1S^n and on the $(2n-1)$-dimensional manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1}, and to describe its associated Legendre submanifolds. This will enable us to study submanifolds of $\mathbf{R}^n$ or S^n within the context of Lie sphere geometry in a natural way. This theory was first developed extensively in a modern setting by Pinkall [40] (see also Cecil-Chern [11]–[12] or the books [9, pp. 51–60], [19, pp. 202–212]).

We consider T_1S^n to be the $(2n-1)$-dimensional submanifold of

$$S^n \times S^n \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$$

given by

$$T_1S^n = \{(x,\xi) \mid \ |x| = 1,\ |\xi| = 1,\ x \cdot \xi = 0\}. \tag{34}$$

As shown in the previous section, the points on a line ℓ lying on Q^{n+1} correspond to the spheres in a parabolic pencil of spheres in S^n. In particular, as in equation (30), ℓ contains one point $[k_1] = [(1,x,0)]$ corresponding to a point sphere in S^n, and one point $[k_2] = [(0,\xi,1)]$ corresponding to a great sphere in S^n, where the coordinates are with respect to the standard orthonormal basis $\{e_1,\ldots,e_{n+3}\}$ of $\mathbf{R}^{n+3}_2$. Thus we get a bijective correspondence between the points (x,ξ) of T_1S^n and the space Λ^{2n-1} of lines on Q^{n+1} given by the map:

$$(x,\xi) \mapsto [Y_1(x,\xi), Y_{n+3}(x,\xi)], \tag{35}$$

where

$$Y_1(x,\xi) = (1,x,0), \quad Y_{n+3}(x,\xi) = (0,\xi,1). \tag{36}$$

We use this correspondence to place a natural differentiable structure on Λ^{2n-1} in such a way as to make the map in equation (35) a diffeomorphism.

We now show how to define a contact structure on the manifold T_1S^n. By the diffeomorphism in equation (35), this also determines a contact structure on Λ^{2n-1}. Recall that a $(2n-1)$-dimensional manifold V^{2n-1} is said to be a *contact manifold* if it carries a globally defined 1-form ω such that

$$\omega \wedge (d\omega)^{n-1} \neq 0 \tag{37}$$

at all points of V^{2n-1}. Such a form ω is called a *contact form*. A contact form ω determines a codimension one distribution (the *contact distribution*) D on V^{2n-1} defined by

$$D_p = \{Y \in T_pV^{2n-1} \mid \omega(Y) = 0\}, \tag{38}$$

for $p \in V^{2n-1}$. This distribution is as far from being integrable as possible, in that there exist integral submanifolds of D of dimension $n-1$ but none of higher dimension (see, for example, [9, p. 57]). The distribution D determines the corresponding contact form ω up to multiplication by a nonvanishing smooth function.

A tangent vector to T_1S^n at a point (x,ξ) can be written in the form (X,Z) where

$$X \cdot x = 0, \quad Z \cdot \xi = 0. \tag{39}$$

Differentiation of the condition $x \cdot \xi = 0$ implies that (X,Z) also satisfies

$$X \cdot \xi + Z \cdot x = 0. \tag{40}$$

Using the method of moving frames, one can show that the form ω defined by

$$\omega(X,Z) = X \cdot \xi, \tag{41}$$

is a contact form on T_1S^n (see, for example, Cecil–Chern [11] or the book [9, pp. 52–56]), and we will omit the proof here.

At a point (x,ξ), the distribution D is the $(2n-2)$-dimensional space of vectors (X,Z) satisfying $X \cdot \xi = 0$, as well as the equations (39) and (40). The equation $X \cdot \xi = 0$ together with equation (40) implies that

$$Z \cdot x = 0, \tag{42}$$

for vectors (X, Z) in D.

Returning to the general theory of contact structures, we let V^{2n-1} be a contact manifold with contact form ω and corresponding contact distribution D, as in equation (38). An immersion $\phi : W^k \to V^{2n-1}$ of a smooth k-dimensional manifold W^k into V^{2n-1} is called an *integral submanifold* of the distribution D if $\phi^*\omega = 0$ on W^k, i.e., for each tangent vector Y at each point $w \in W$, the vector $d\phi(Y)$ is in the distribution D at the point $\phi(w)$. (See Blair [3, p. 36].) It is well known (see, for example, [9, p. 57]) that the contact distribution D has integral submanifolds of dimension $n-1$, but none of higher dimension. These integral submanifolds of maximal dimension are called *Legendre submanifolds* of the contact structure.

In our specific case, we now formulate conditions for a smooth map $\mu : M^{n-1} \to T_1S^n$ to be a Legendre submanifold. We consider T_1S^n as a submanifold of $S^n \times S^n$ as in equation (34), and so we can write $\mu = (f, \xi)$, where f and ξ are both smooth maps from M^{n-1} to S^n. We have the following theorem (see [9, p. 58]) giving necessary and sufficient conditions for μ to be a Legendre submanifold.

Theorem 6.1. *A smooth map $\mu = (f, \xi)$ from an $(n-1)$-dimensional manifold M^{n-1} into T_1S^n is a Legendre submanifold if and only if the following three conditions are satisfied.*

(1) *Scalar product conditions:* $f \cdot f = 1, \quad \xi \cdot \xi = 1, \quad f \cdot \xi = 0.$

(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $df(X)$ and $d\xi(X)$ are both equal to zero.*

(3) *Contact condition:* $df \cdot \xi = 0$.

Note that by equation (34), the scalar product conditions are precisely the conditions necessary for the image of the map $\mu = (f, \xi)$ to be contained in T_1S^n. Next, since $d\mu(X) = (df(X), d\xi(X))$, Condition (2) is necessary and sufficient for μ to be an immersion. Finally, from equation (41), we see that $\omega(d\mu(X)) = df(X) \cdot \xi(x)$, for each $X \in T_xM^{n-1}$. Hence Condition (3) is equivalent to the requirement that $\mu^*\omega = 0$ on M^{n-1}.

We now want to translate these conditions into the projective setting, and find necessary and sufficient conditions for a smooth map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ to be a Legendre submanifold. We again make use of the diffeomorphism defined in equation (35) between T_1S^n and Λ^{2n-1}.

For each $x \in M^{n-1}$, we know that $\lambda(x)$ is a line on the quadric Q^{n+1}. This line contains exactly one point $[Y_1(x)] = [(1, f(x), 0)]$ corresponding to a point sphere in S^n, and one point $[Y_{n+3}(x)] = [(0, \xi(x), 1)]$ corresponding to a great sphere in S^n. These two formulas define maps f and ξ from M^{n-1} to S^n which depend on the choice of orthonormal basis $\{e_1, \ldots, e_{n+2}\}$ for the orthogonal complement of e_{n+3}.

The map $[Y_1]$ from M^{n-1} to Q^{n+1} is called the *Möbius projection* or *point sphere map* of λ, and the map $[Y_{n+3}]$ from M^{n-1} to Q^{n+1} is called the *great sphere map*. The maps f and ξ are called the *spherical projection* of λ, and the *spherical field of unit normals* of λ, respectively.

In this way, λ determines a map $\mu = (f, \xi)$ from M^{n-1} to T_1S^n, and because of the diffeomorphism (35), λ is a Legendre submanifold if and only if μ satisfies the conditions of Theorem 6.1.

It is often useful to have conditions for when λ determines a Legendre submanifold that do not depend on the special parametrization of λ in terms of the point sphere and great sphere maps, $[Y_1]$ and $[Y_{n+3}]$. In fact, in many applications of Lie sphere geometry to submanifolds of S^n or $\mathbf{R}^n$, it is better to consider $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are not the point sphere and great sphere maps.

Pinkall [40] gave the following projective formulation of the conditions needed for a Legendre submanifold. In his paper, Pinkall referred to a Legendre submanifold as a "Lie geometric hypersurface." The proof that the three conditions of the theorem below are equivalent to the three conditions of Theorem 6.1 can be found in [9, pp. 59–60].

Theorem 6.2. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a smooth map with $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are smooth maps from M^{n-1} into $\mathbf{R}^{n+3}_2$. Then λ determines a Legendre submanifold if and only if Z_1 and Z_{n+3} satisfy the following conditions.*

(1) *Scalar product conditions: for each $x \in M^{n-1}$, the vectors $Z_1(x)$ and $Z_{n+3}(x)$ are linearly independent and*

$$\langle Z_1, Z_1 \rangle = 0, \quad \langle Z_{n+3}, Z_{n+3} \rangle = 0, \quad \langle Z_1, Z_{n+3} \rangle = 0.$$

(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $dZ_1(X)$ and $dZ_{n+3}(X)$ are both in*

$$\text{Span } \{Z_1(x), Z_{n+3}(x)\}.$$

(3) *Contact condition:* $\langle dZ_1, Z_{n+3}\rangle = 0$.

These conditions are invariant under a reparametrization $\lambda = [W_1, W_{n+3}]$, *where* $W_1 = \alpha Z_1 + \beta Z_{n+3}$ *and* $W_{n+3} = \gamma Z_1 + \delta Z_{n+3}$, *for smooth functions* $\alpha, \beta, \gamma, \delta$ *on* M^{n-1} *with* $\alpha\delta - \beta\gamma \neq 0$.

Every oriented hypersurface in S^n or $\mathbf{R}^n$ naturally induces a Legendre submanifold of Λ^{2n-1}, as does every submanifold of codimension $m > 1$ in these spaces. Conversely, a Legendre submanifold naturally induces a smooth map into S^n or $\mathbf{R}^n$, which may have singularities. We now study the details of these maps.

Let $f : M^{n-1} \to S^n$ be an immersed oriented hypersurface with field of unit normals $\xi : M^{n-1} \to S^n$. The induced Legendre submanifold is given by the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda(x) = [Y_1(x), Y_{n+3}(x)]$, where

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+3}(x) = (0, \xi(x), 1). \tag{43}$$

The map λ is called the *Legendre lift* of the immersion f with field of unit normals ξ.

To show that λ is a Legendre submanifold, we check the conditions of Theorem 6.2. Condition (1) is satisfied since both f and ξ are maps into S^n, and $\xi(x)$ is tangent to S^n at $f(x)$ for each x in M^{n-1}. Since f is an immersion, $dY_1(X) = (0, df(X), 0)$ is not in Span $\{Y_1(x), Y_{n+3}(x)\}$, for any nonzero vector $X \in T_xM^{n-1}$, and so Condition (2) is satisfied. Finally, Condition (3) is satisfied since

$$\langle dY_1(X), Y_{n+3}(x)\rangle = df(X) \cdot \xi(x) = 0,$$

because ξ is a field of unit normals to f.

In the case of a submanifold $\phi : V \to S^n$ of codimension $m+1$ greater than one, the domain of the Legendre lift is be the unit normal bundle B^{n-1} of the submanifold $\phi(V)$. We consider B^{n-1} to be the submanifold of $V \times S^n$ given by

$$B^{n-1} = \{(x,\xi) | \phi(x) \cdot \xi = 0,\ d\phi(X) \cdot \xi = 0,\ \text{for all } X \in T_xV\}.$$

The *Legendre lift* of ϕ is the map $\lambda : B^{n-1} \to \Lambda^{2n-1}$ defined by

$$\lambda(x,\xi) = [Y_1(x,\xi), Y_{n+3}(x,\xi)], \tag{44}$$

where

$$Y_1(x,\xi) = (1, \phi(x), 0), \quad Y_{n+3}(x,\xi) = (0, \xi, 1). \tag{45}$$

Geometrically, $\lambda(x,\xi)$ is the line on the quadric Q^{n+1} corresponding to the parabolic parabolic pencil of spheres in S^n in oriented contact at the contact element $(\phi(x), \xi) \in T_1S^n$. In [9, pp. 61–62], we show that λ satisfies the conditions of Theorem 6.2, and we omit the proof here.

Similarly, suppose that $F : M^{n-1} \to \mathbf{R}^n$ is an oriented hypersurface with field of unit normals $\eta : M^{n-1} \to \mathbf{R}^n$, where we identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^{n+3}_2$ spanned by $\{e_3, \ldots, e_{n+2}\}$. The Legendre lift of (F, η) is the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Y_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{46}$$

By equation (21), $[Y_1(x)]$ corresponds to the point sphere and $[Y_{n+3}(x)]$ corresponds to the hyperplane in the parabolic pencil determined by the line $\lambda(x)$ for each $x \in M^{n-1}$. One can easily verify that Conditions (1)–(3) of Theorem 6.2 are satisfied in a manner similar to the spherical case. In the case of a submanifold $\psi : V \to \mathbf{R}^n$ of codimension greater than one, the Legendre lift of ψ is the map λ from the unit normal bundle B^{n-1} to Λ^{2n-1} defined by $\lambda(x,\eta) = [Y_1(x,\eta), Y_{n+3}(x,\eta)]$, where

$$\begin{aligned} Y_1(x,\eta) &= (1 + \psi(x)\cdot\psi(x), 1 - \psi(x)\cdot\psi(x), 2\psi(x), 0)/2, \\ Y_{n+3}(x,\eta) &= (\psi(x)\cdot\eta, -(\psi(x)\cdot\eta), \eta, 1). \end{aligned} \tag{47}$$

The verification that the pair $\{Y_1, Y_{n+3}\}$ satisfies conditions (1)–(3) of Theorem 6.2 is similar to that for submanifolds of S^n of codimension greater than one, and we omit that proof here also.

Conversely, suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is an arbitrary Legendre submanifold. We have seen above that we can parametrize λ as $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{48}$$

for the spherical projection f and spherical field of unit normals ξ. Both f and ξ are smooth maps, but neither need be an immersion or even have constant rank (see [9, pp. 63–64]).

The Legendre lift of an oriented hypersurface in S^n is the special case where the spherical projection f is an immersion, i.e., f has constant rank $n-1$ on M^{n-1}. In the case of the Legendre lift of a submanifold $\phi : V^k \to$

S^n, the spherical projection $f : B^{n-1} \to S^n$ defined by $f(x,\xi) = \phi(x)$ has constant rank k.

If the range of the point sphere map $[Y_1]$ does not contain the improper point $[(1,-1,0,\ldots,0)]$, then λ also determines a *Euclidean projection* F, where $F : M^{n-1} \to \mathbf{R}^n$, and a *Euclidean field of unit normals* η, where $\eta : M^{n-1} \to \mathbf{R}^n$. These are defined by the equation $\lambda = [Z_1, Z_{n+3}]$, where

$$Z_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Z_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{49}$$

Here $[Z_1(x)]$ corresponds to the unique point sphere in the parabolic pencil determined by $\lambda(x)$, and $[Z_{n+3}(x)]$ corresponds to the unique plane in this pencil. As in the spherical case, the smooth maps F and η need not have constant rank.

7 Curvature Spheres

To motivate the definition of a curvature sphere we consider the case of an oriented hypersurface $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$. (We could consider an oriented hypersurface in $\mathbf{R}^n$, but the calculations are simpler in the spherical case.)

The shape operator of f at a point $x \in M^{n-1}$ is the symmetric linear transformation $A : T_xM^{n-1} \to T_xM^{n-1}$ defined on the tangent space T_xM^{n-1} by the equation

$$df(AX) = -d\xi(X), \quad X \in T_xM^{n-1}. \tag{50}$$

The eigenvalues of A are called the *principal curvatures*, and the corresponding eigenvectors are called the *principal vectors*. We next recall the notion of a focal point of an immersion. For each real number t, define a map

$$f_t : M^{n-1} \to S^n,$$

by

$$f_t = \cos t \ f + \sin t \ \xi. \tag{51}$$

For each $x \in M^{n-1}$, the point $f_t(x)$ lies an oriented distance t along the normal geodesic to $f(M^{n-1})$ at $f(x)$. A point $p = f_t(x)$ is called a *focal point of multiplicity* $m > 0$ *of* f *at* x if the nullity of df_t is equal to m at x. Geometrically, one thinks of focal points as points where nearby normal geodesics intersect. It is well known that the location of focal points is related to the principal curvatures. Specifically, if $X \in T_x M^{n-1}$, then by equation (50) we have

$$df_t(X) = \cos t \; df(X) + \sin t \; d\xi(X) = df(\cos t \; X - \sin t \; AX). \qquad (52)$$

Thus, $df_t(X)$ equals zero for $X \neq 0$ if and only if $\cot t$ is a principal curvature of f at x, and X is a corresponding principal vector. Hence, $p = f_t(x)$ is a focal point of f at x of multiplicity m if and only if $\cot t$ is a principal curvature of multiplicity m at x. Note that each principal curvature

$$\kappa = \cot t, \quad 0 < t < \pi,$$

produces two distinct antipodal focal points on the normal geodesic with parameter values t and $t + \pi$. The oriented hypersphere centered at a focal point p and in oriented contact with $f(M^{n-1})$ at $f(x)$ is called a *curvature sphere* of f at x. The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal curvatures and curvature spheres is bijective. The multiplicity of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature.

We now formulate the notion of curvature sphere in the context of Lie sphere geometry. As in equation (43), the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of the oriented hypersurface (f, ξ) is given by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \qquad (53)$$

For each $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be parametrized as

$$[K_t(x)] = [\cos t \; Y_1(x) + \sin t \; Y_{n+3}(x)] = [(\cos t, f_t(x), \sin t)], \qquad (54)$$

where f_t is given in equation (51) above. By equation (25), the point $[K_t(x)]$ in Q^{n+1} corresponds to the oriented sphere in S^n with center $f_t(x)$ and signed radius t. This sphere is in oriented contact with the oriented hypersurface $f(M^{n-1})$ at $f(x)$. Given a tangent vector $X \in T_x M^{n-1}$, we have

$$dK_t(X) = (0, df_t(X), 0). \qquad (55)$$

Thus, $dK_t(X) = (0, 0, 0)$ for a nonzero vector $X \in T_x M^{n-1}$ if and only if $df_t(X) = 0$, i.e., $p = f_t(x)$ is a focal point of f at x corresponding to the principal curvature $\cot t$. The vector X is a principal vector corresponding to the principal curvature $\cot t$, and it is also called a principal vector corresponding to the curvature sphere $[K_t]$.

This characterization of curvature spheres depends on the parametrization of $\lambda = [Y_1, Y_{n+3}]$ given by the point sphere and great sphere maps $[Y_1]$ and $[Y_{n+3}]$, respectively, and it has only been defined in the case where the spherical projection f is an immersion. We now give a projective formulation of the definition of a curvature sphere that is independent of the parametrization of λ and is valid for an arbitrary Legendre submanifold.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by the pair $\{Z_1, Z_{n+3}\}$, as in Theorem 6.2. Let $x \in M^{n-1}$ and $r, s \in \mathbf{R}$ with at least one of r and s not equal to zero. The sphere,

$$[K] = [rZ_1(x) + sZ_{n+3}(x)],$$

is called a *curvature sphere* of λ at x if there exists a nonzero vector X in T_xM^{n-1} such that

$$r\ dZ_1(X) + s\ dZ_{n+3}(X) \in \text{Span}\ \{Z_1(x), Z_{n+3}(x)\}. \tag{56}$$

The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$. This definition is invariant under a change of parametrization of the form considered in the statement of Theorem 6.2. Furthermore, if we take the special parametrization $Z_1 = Y_1$, $Z_{n+3} = Y_{n+3}$ given in equation (53), then condition (56) holds if and only if $r\ dY_1(X) + s\ dY_{n+3}(X)$ actually equals $(0, 0, 0)$.

From equation (56), it is clear that the set of principal vectors corresponding to a given curvature sphere $[K]$ at x is a subspace of T_xM^{n-1}. This set is called the *principal space* corresponding to the curvature sphere $[K]$. Its dimension is the *multiplicity* of $[K]$. The reader is referred to Cecil–Chern [11]–[12] for a development of the notion of a curvature sphere in the context of Lie sphere geometry without any reference to submanifolds of S^n or $\mathbf{R}^n$.

We next show that a Lie sphere transformation maps curvature spheres to curvature spheres. We first need to discuss the notion of Lie equivalent Legendre submanifolds. Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_{n+3}]$. Suppose $\beta = P(B)$ is the Lie sphere transformation induced by an orthogonal transformation B in the group $O(n+1, 2)$. Since B is orthogonal, the maps, $W_1 = BZ_1$, $W_{n+3} = BZ_{n+3}$, satisfy the Conditions (1)–(3) of Theorem 6.2, and thus $\gamma = [W_1, W_{n+3}]$ is a Legendre submanifold which we denote by $\beta\lambda : M^{n-1} \to \Lambda^{2n-1}$. We say that the Legendre submanifolds λ and $\beta\lambda$ are *Lie equivalent*. In terms of submanifolds of real space forms, we say that two immersed submanifolds of $\mathbf{R}^n$ or S^n are *Lie equivalent* if their Legendre lifts are Lie equivalent.

Theorem 7.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. The point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\beta[K]$ is a curvature sphere of the Legendre submanifold $\beta\lambda$ at x. Furthermore, the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical.*

Proof. Let $\lambda = [Z_1, Z_{n+3}]$ and $\beta\lambda = [W_1, W_{n+3}]$ as above. For a tangent vector $X \in T_x M^{n-1}$ and real numbers r and s, at least one of which is not zero, we have

$$\begin{aligned} r\ dW_1(X) + s\ dW_{n+3}(X) &= r\ d(BZ_1)(X) + s\ d(BZ_{n+3})(X) \\ &= B(r\ dZ_1(X) + s\ dZ_{n+3}(X)), \end{aligned} \tag{57}$$

since B is a constant linear transformation. Thus, we see that

$$r\ dW_1(X) + s\ dW_{n+3}(X) \in \text{Span}\ \{W_1(x), W_{n+3}(x)\}$$

if and only if

$$r\ dZ_1(X) + s\ dZ_{n+3}(X) \in \text{Span}\ \{Z_1(x), Z_{n+3}(x)\}.$$

□

We next consider the case when the Lie sphere transformation β is a spherical parallel transformation P_t defined in terms of the standard basis of $\mathbf{R}^{n+3}_2$ by

$$\begin{aligned} P_t e_1 &= \cos t\ e_1 + \sin t\ e_{n+3}, \\ P_t e_{n+3} &= -\sin t\ e_1 + \cos t\ e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \le i \le n+2. \end{aligned} \tag{58}$$

The transformation P_t has the effect of adding t to the signed radius of each oriented sphere in S^n while keeping the center fixed (see, for example, [9, pp. 48–49]).

If $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold parametrized by the point sphere map $Y_1 = (1, f, 0)$ and the great sphere map $Y_{n+3} = (0, \xi, 1)$, then $P_t\lambda = [W_1, W_{n+3}]$, where

$$W_1 = P_t Y_1 = (\cos t, f, \sin t), \quad W_{n+3} = P_t Y_{n+3} = (-\sin t, \xi, \cos t). \tag{59}$$

Note that W_1 and W_{n+3} are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_{n+3} of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t\; W_1 - \sin t\; W_{n+3} = (1, \cos t\; f - \sin t\; \xi, 0), \\ Z_{n+3} &= \sin t\; W_1 + \cos t\; W_{n+3} = (0, \sin t\; f + \cos t\; \xi, 1). \end{aligned} \tag{60}$$

From this, we see that $P_t\lambda$ has spherical projection and spherical unit normal field given, respectively, by

$$\begin{aligned} f_{-t} &= \cos t\; f - \sin t\; \xi = \cos(-t) f + \sin(-t)\xi, \\ \xi_{-t} &= \sin t\; f + \cos t\; \xi = -\sin(-t) f + \cos(-t)\xi. \end{aligned} \tag{61}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ. Formula (61) shows the close correspondence between these parallel submanifolds and the parallel hypersurfaces f_t to f, in the case where f is an immersed hypersurface.

In the case where the spherical projection f is an immersion at a point $x \in M^{n-1}$, we know that the number of values of t in the interval $[0, \pi)$ for which f_t is not an immersion is at most $n - 1$, the maximum number of distinct principal curvatures of f at x. Pinkall [40, p. 428] proved that this statement is also true for an arbitrary Legendre submanifold, even if the spherical projection f is not an immersion at x by proving the following theorem (see also [9, pp. 68–72] for a proof).

Theorem 7.2. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with spherical projection f and spherical unit normal field ξ. Then for each $x \in M^{n-1}$, the parallel map,*

$$f_t = \cos t\; f + \sin t\; \xi,$$

fails to be an immersion at x for at most $n - 1$ values of $t \in [0, \pi)$.

As a consequence of Pinkall's theorem, one can pass to a parallel submanifold, if necessary, to obtain the following important corollary by using well known results concerning immersed hypersurfaces in S^n. Note that parts (a)–(c) of the corollary are pointwise statements, while (d)–(e) hold on an open set U if they can be shown to hold in a neighborhood of each point of U.

Corollary 7.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Then:*

(a) *at each point $x \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $K_1, \ldots, K_g$,*

(b) *the principal vectors corresponding to a curvature sphere K_i form a subspace T_i of the tangent space $T_x M^{n-1}$,*

(c) *the tangent space $T_x M^{n-1} = T_1 \oplus \cdots \oplus T_g$,*

(d) *if the dimension of a given T_i is constant on an open subset U of M^{n-1}, then the principal distribution T_i is integrable on U,*

(e) *if $\dim T_i = m > 1$ on an open subset U of M^{n-1}, then the curvature sphere map K_i is constant along the leaves of the principal foliation T_i.*

We can also generalize the notion of a curvature surface for hypersurfaces in real space forms to Legendre submanifolds. Specifically, let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. A connected submanifold S of M^{n-1} is called a *curvature surface* if at each $x \in S$, the tangent space $T_x S$ is equal to some principal space T_i. For example, if $\dim T_i$ is constant on an open subset U of M^{n-1}, then each leaf of the principal foliation T_i is a curvature surface on U. It is also possible to have a curvature surface S which is not a leaf of a principal foliation (see [9, p. 69] for an example).

8 Dupin Submanifolds

Next we generalize the definition of a Dupin hypersurface in a real space form to the setting of Legendre submanifolds in Lie sphere geometry. We say that a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a *Dupin submanifold* if:

(a) along each curvature surface, the corresponding curvature sphere map is constant.

The Dupin submanifold λ is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) the number g of distinct curvature spheres is constant on M.

In the case of the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of an immersed Dupin hypersurface $f : M^{n-1} \to S^n$, the submanifold λ is a Dupin submanifold, since a curvature sphere map of λ is constant along a curvature surface if and only if the corresponding principal curvature map of f is constant along that curvature surface. Similarly, λ is proper Dupin if and only if f is proper Dupin, since the number of distinct curvatures spheres of λ at a point $x \in M^{n-1}$ equals the number of distinct principal curvatures of f at x. Particularly important examples of proper Dupin submanifolds are the the Legendre lifts of isoparametric hypersurfaces in S^n.

We now show that Theorem 7.1 implies that both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations. Many important classification results for Dupin submanifolds have been obtained in the setting of Lie sphere geometry (see Chapter 5 of [9]).

Theorem 8.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation.*

(a) *If λ is Dupin, then $\beta\lambda$ is Dupin.*

(b) *If λ is proper Dupin, then $\beta\lambda$ is proper Dupin.*

Proof. By Theorem 7.1, a point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at $x \in M$ if and only if the point $\beta[K]$ is a curvature sphere of $\beta\lambda$ at x, and the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical. Since these principal spaces are the same, if S is a curvature surface of λ corresponding to a curvature sphere map $[K]$, then S is also a curvature surface of $\beta\lambda$ corresponding to a curvature sphere map $\beta[K]$, and clearly $[K]$ is constant along S if and only if $\beta[K]$ is constant along S. This proves part (a) of the theorem. Part (b) also follows immediately from Theorem 7.1, since for each $x \in M$, the number g of distinct curvature spheres of λ at x equals the number of distinct curvatures spheres of $\beta\lambda$ at x. So if this number g is constant on M for λ, then it is constant on M for $\beta\lambda$. □

9 Principal Curvatures and Curvature Spheres

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. As before, we can write $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{62}$$

where f and ξ are the spherical projection and spherical field of unit normals, respectively.

For $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be written in the form,

$$\mu Y_1(x) + Y_{n+3}(x), \tag{63}$$

that is, we take μ as an inhomogeneous coordinate along the projective line $\lambda(x)$. Then the point sphere $[Y_1]$ corresponds to $\mu = \infty$. The next two theorems give the relationship between the coordinates of the curvature spheres of λ and the principal curvatures of f, in the case where f has constant rank. In the first theorem, we assume that the spherical projection f is an immersion on M^{n-1}. By Theorem 7.2, we know that this can always be achieved locally by passing to a parallel submanifold.

Theorem 9.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold whose spherical projection $f : M^{n-1} \to S^n$ is an immersion. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (62). Then the curvature spheres of λ at a point $x \in M^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_g$ are the distinct principal curvatures at x of the oriented hypersurface f with field of unit normals ξ. The multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i.

Proof. Let X be a nonzero vector in $T_x M^{n-1}$. Then for any real number μ,

$$d(\mu Y_1 + Y_{n+3})(X) = (0, \mu\, df(X) + d\xi(X), 0).$$

This vector is in Span $\{Y_1(x), Y_{n+3}(x)\}$ if and only if

$$\mu\, df(X) + d\xi(X) = 0,$$

i.e., μ is a principal curvature of f with corresponding principal vector X. $\square$

We next consider the case where the point sphere map Y_1 is a curvature sphere of constant multiplicity m on M^{n-1}. By Corollary 7.1, the corresponding principal distribution is a foliation, and the curvature sphere map $[Y_1]$ is constant along the leaves of this foliation. Thus the map $[Y_1]$ factors through an immersion $[W_1]$ from the space of leaves V of this foliation into Q^{n+1}. We can write $[W_1] = [(1, \phi, 0)]$, where $\phi : V \rightarrow S^n$ is an immersed submanifold of codimension $m + 1$. The manifold M^{n-1} is locally diffeomorphic to an open subset of the unit normal bundle B^{n-1} of the submanifold ϕ, and λ is essentially the Legendre lift of $\phi(V)$, as defined in Section 6. The following theorem relates the curvature spheres of λ to the principal curvatures of ϕ. Recall that the point sphere and great sphere maps for λ are given as in equation (45) by

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{64}$$

Theorem 9.2. *Let $\lambda : B^{n-1} \rightarrow \Lambda^{2n-1}$ be the Legendre lift of an immersed submanifold $\phi(V)$ in S^n of codimension $m+1$. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (64). Then the curvature spheres of λ at a point $(x, \xi) \in B^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_{g-1}$ are the distinct principal curvatures of the shape operator A_ξ, and $\kappa_g = \infty$. For $1 \leq i \leq g-1$, the multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i, while the multiplicity of $[K_g]$ is m.

The proof of this theorem is similar to that of Theorem 9.1, but one must introduce local coordinates on the unit normal bundle to get a complete proof (see [9, p. 74]).

We close this section with a local Lie geometric characterization of Legendre submanifolds that are Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n (see Cecil [7] or [9, p. 77]). Here a line in P^{n+2} is called *timelike* if it contains only timelike points. This means that an orthonormal basis for the 2-plane in $\mathbf{R}^{n+3}_2$ determined by the timelike line consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$. This theorem has been useful in obtaining various classification results for proper Dupin hypersurfaces.

Theorem 9.3. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $[K_1], \dots, [K_g]$ at each point. Then λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n if and only if there exist g points $[P_1], \dots, [P_g]$ on a timelike line in P^{n+2} such that*

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Proof. If λ is the Legendre lift of an isoparametric hypersurface in S^n, then all the spheres in a family $[K_i]$ have the same radius ρ_i, where $0 < \rho_i < \pi$. By formula (25), this is equivalent to the condition $\langle K_i, P_i \rangle = 0$, where

$$P_i = \sin \rho_i \; e_1 - \cos \rho_i \; e_{n+3}, \quad 1 \leq i \leq g, \tag{65}$$

are g points on the timelike line $[e_1, e_{n+3}]$. Since a Lie sphere transformation preserves curvature spheres, timelike lines and the polarity relationship, the same is true for any image of λ under a Lie sphere transformation.

Conversely, suppose that there exist g points $[P_1], \dots, [P_g]$ on a timelike line ℓ such that $\langle K_i, P_i \rangle = 0$, for $1 \leq i \leq g$. Let β be a Lie sphere transformation that maps ℓ to the line $[e_1, e_{n+3}]$. Then the curvature spheres $\beta[K_i]$ of $\beta\lambda$ are orthogonal to the points $[Q_i] = \beta[P_i]$ on the line $[e_1, e_{n+3}]$. This means that the spheres corresponding to $\beta[K_i]$ have constant radius on M^{n-1}. By applying a parallel transformation P_t, if necessary, we can arrange that none of these curvature spheres has radius zero. Then $P_t \beta \lambda$ is the Legendre lift of an isoparametric hypersurface in S^n. □

10 Cyclides of Dupin

We now turn our attention to Pinkall's classification of the cyclides of Dupin of arbitrary dimension, obtained by using the methods of Lie sphere geometry. Our presentation here is based on the accounts of this subject given in [9, pp. 148–159] and [19, pp. 263–283]. A proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with two distinct curvature spheres of respective multiplicities p and q at each point is called a *cyclide of Dupin of characteristic* (p, q).

We will prove that any connected cyclide of Dupin of characteristic (p, q) is contained in a unique compact, connected cyclide of Dupin of characteristic (p, q). Furthermore, every compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \tag{66}$$

where p and q are positive integers such that $p + q = n - 1$. Thus any two compact, connected cyclides of Dupin of the same characteristic are Lie equivalent.

It is well known that the product $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is an isoparametric hypersurface in S^n with two distinct principal curvatures having multiplicities $m_1 = p$ and $m_2 = q$ (see, for example, [19, pp. 110–111]). Furthermore, every compact isoparametric hypersurface in S^n with two principal curvatures of multiplicities p and q is Lie equivalent to $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$, since it is congruent to a parallel hypersurface of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$.

Although $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is a good model for the cyclides, it is often easier to work with the two focal submanifolds $S^q(1) \times \{0\}$ and $\{0\} \times S^p(1)$ in proving classification results. The Legendre lifts of these two focal submanifolds are Lie equivalent to the Legendre lift of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$, since they are parallel submanifolds of the Legendre lift of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$. In fact, the hypersurface $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is a tube of spherical radius $\pi/4$ in S^n over either of its two focal submanifolds.

We now describe our standard model of a cyclide of characteristic (p, q) in the context of Lie sphere geometry, as in Pinkall's paper [40] (see also [9, p. 149]). Let $\{e_1, \ldots, e_{n+3}\}$ be the standard orthonormal basis for $\mathbf{R}^{n+3}_2$, with e_1 and e_{n+3} unit timelike vectors, and $\{e_2, \ldots, e_{n+2}\}$ unit spacelike vectors. Then S^n is the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ spanned by $\{e_2, \ldots, e_{n+2}\}$. Let

$$\Omega = \text{Span}\ \{e_1, \ldots, e_{q+2}\}, \quad \Omega^\perp = \text{Span}\ \{e_{q+3}, \ldots, e_{n+3}\}. \tag{67}$$

These spaces have signatures $(q + 1, 1)$ and $(p + 1, 1)$, respectively. The intersection $\Omega \cap Q^{n+1}$ is the quadric given in homogeneous coordinates by

$$x_1^2 = x_2^2 + \cdots + x_{q+2}^2, \quad x_{q+3} = \cdots = x_{n+3} = 0. \tag{68}$$

This set is diffeomorphic to the unit sphere S^q in

$$\mathbf{R}^{q+1} = \text{Span}\ \{e_2, \ldots, e_{q+2}\},$$

by the diffeomorphism $\phi : S^q \to \Omega \cap Q^{n+1}$, defined by $\phi(v) = [e_1 + v]$. Similarly, the quadric $\Omega^\perp \cap Q^{n+1}$ is diffeomorphic to the unit sphere S^p in

$$\mathbf{R}^{p+1} = \text{Span}\ \{e_{q+3}, \ldots, e_{n+2}\}$$

by the diffeomorphism $\psi : S^p \to \Omega^\perp \cap Q^{n+1}$ defined by $\psi(u) = [u + e_{n+3}]$. The model that we will use for the cyclides in Lie sphere geometry is the Legendre submanifold $\lambda : S^p \times S^q \to \Lambda^{2n-1}$ defined by

$$\lambda(u, v) = [k_1, k_2], \text{with } [k_1(u, v)] = [\phi(v)], \quad [k_2(u, v)] = [\psi(u)]. \tag{69}$$

It is easy to check that the Legendre Conditions (1)–(3) of Theorem 6.2 are satisfied by the pair $\{k_1, k_2\}$. To find the curvature spheres of λ, we decompose the tangent space to $S^p \times S^q$ at a point (u, v) as

$$T_{(u,v)} S^p \times S^q = T_u S^p \times T_v S^q.$$

Then $dk_1(X, 0) = 0$ for all $X \in T_u S^p$, and $dk_2(Y) = 0$ for all Y in $T_v S^q$. Thus, $[k_1]$ and $[k_2]$ are curvature spheres of λ with respective multiplicities p and q. Furthermore, the image of $[k_1]$ lies in the quadric $\Omega \cap Q^{n+1}$, and the image of $[k_2]$ is contained in the quadric $\Omega^\perp \cap Q^{n+1}$. The point sphere map of λ is $[k_1]$, and thus λ is the Legendre lift of the focal submanifold $S^q \times \{0\} \subset S^n$, considered as a submanifold of codimension $p + 1$ in S^n. As noted above, this Legendre lift λ of the focal submanifold is Lie equivalent to the Legendre lift of the standard product of spheres by means of a parallel transformation.

We now prove Pinkall's [40] classification of proper Dupin submanifolds with two distinct curvature spheres at each point. Pinkall's proof depends on establishing the existence of a local principal coordinate system. This can always be done in the case of $g = 2$ curvature spheres, because the the sum of the dimensions of the two principal spaces is $n-1$, the dimension of M^{n-1} (see, for example, [19, p. 249]). Such a coordinate system might not exist in the case $g > 2$. In fact, if M is an isoparametric hypersurface in S^n with more than two distinct principal curvatures, then there cannot exist a local principal coordinate system on M (see, for example, [18, pp. 180–184] or [19, pp. 248–252]).

For a different proof of Pinkall's theorem (Theorem 10.1 below) using the method of moving frames, see the paper of Cecil-Chern [12] or [19, pp. 266–273]. That approach generalizes to the study of proper Dupin hypersurfaces with $g > 2$ curvature spheres (see, for example, Cecil and Jensen [14], [15]).

Note that before Pinkall's paper, Cecil and Ryan [16]–[17] (see also [18, pp. 166–179]) proved a classification of complete cyclides in $\mathbf{R}^n$ using techniques of Euclidean submanifold theory. However, the proof used the assumption of completeness in an essential way, and that theorem did not contain part (a) of Pinkall's Theorem 10.1 below.

Theorem 10.1. (a) *Every connected cyclide of Dupin of characteristic (p,q) is contained in a unique compact, connected cyclide of Dupin characteristic (p,q).*

(b) *Every compact, connected cyclide of Dupin of characteristic (p,q) is Lie equivalent to the Legendre lift of a standard product of two spheres*

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \qquad (70)$$

where $p+q=n-1$. Thus, any two compact, connected cyclides of Dupin of characteristic (p,q) are Lie equivalent.

Proof. Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a connected cyclide of Dupin of characteristic (p,q) with $p+q=n-1$. We may take $\lambda = [k_1, k_2]$, where $[k_1]$ and $[k_2]$ are the curvature spheres with respective multiplicities p and q. Each curvature sphere map factors through an immersion of the space of leaves of its principal foliation T_i for $i=1,2$. Since the sum of the dimensions of T_1 and T_2 equals the dimension of M^{n-1}, locally we can take a principal coordinate system (u,v) (see, for example, [19, p. 249]) defined on an open set

$$W = U \times V \subset \mathbf{R}^p \times \mathbf{R}^q,$$

such that

(i) $[k_1]$ depends only on v, and $[k_2]$ depends only on u, for all $(u,v) \in W$.

(ii) $[k_1(W)]$ and $[k_2(W)]$ are submanifolds of Q^{n+1} of dimensions q and p, respectively.

Now let (u,v) and $(\bar{u},\bar{v})$ be any two points in W. From (i), we have the following key equation,

$$\langle k_1(u,v), k_2(\bar{u},\bar{v})\rangle = \langle k_1(v), k_2(\bar{u})\rangle = \langle k_1(\bar{u},v), k_2(\bar{u},v)\rangle = 0, \qquad (71)$$

since $[k_1]$ and $[k_2]$ are orthogonal at every point $x \in M^{n-1}$, in particular $x = (\bar{u}, v)$.

Let E be the smallest linear subspace of P^{n+2} containing the q-dimensional submanifold $[k_1(W)]$. By equation (71), we have

$$[k_1(W)] \subset E \cap Q^{n+1}, \quad [k_2(W)] \subset E^{\perp} \cap Q^{n+1}. \qquad (72)$$

The dimensions of E and $E^\perp$ as subspaces of P^{n+2} satisfy

$$\dim E + \dim E^\perp = n + 1 = p + q + 2. \tag{73}$$

We claim that $\dim E = q + 1$ and $\dim E^\perp = p + 1$.

To see this, suppose first that $\dim E > q + 1$. Then $\dim E^\perp \leq p$, and $E^\perp \cap Q^{n+1}$ cannot contain the p-dimensional submanifold $k_2(W)$, contradicting equation (72). Similarly, assuming that $\dim E^\perp > p + 1$ leads to a contradiction, since then $\dim E \leq q$, and $E \cap Q^{n+1}$ cannot contain the q-dimensional submanifold $k_1(W)$.

Thus we have

$$\dim E \leq q + 1, \quad \dim E^\perp \leq p + 1.$$

This and equation (73) imply that $\dim E = q + 1$ and $\dim E^\perp = p + 1$. Furthermore, from the fact that $E \cap Q^{n+1}$ and $E^\perp \cap Q^{n+1}$ contain submanifolds of dimensions q and p, respectively, it is easy to deduce that the Lie inner product $\langle\ ,\ \rangle$ has signature $(q + 1, 1)$ on E and $(p + 1, 1)$ on $E^\perp$.

Take an orthornormal basis $\{w_1, \ldots, w_{n+3}\}$ of $\mathbf{R}^{n+3}_2$ with w_1 and w_{n+3} timelike such that

$$E = \text{Span}\ \{w_1, \ldots, w_{q+2}\}, \quad E^\perp = \text{Span}\ \{w_{q+3}, \ldots, w_{n+3}\}. \tag{74}$$

Then $E \cap Q^{n+1}$ is given in homogeneous coordinates $(x_1, \ldots, x_{n+3})$ with respect to this basis by

$$x_1^2 = x_2^2 + \cdots + x_{q+2}^2, \quad x_{q+3} = \cdots = x_{n+3} = 0. \tag{75}$$

This quadric is diffeomorphic to the unit sphere S^q in the span $\mathbf{R}^{q+1}$ of the spacelike vectors $w_2, \ldots, w_{q+2}$ with the diffeomorphism $\gamma : S^q \to E \cap Q^{n+1}$ given by

$$\gamma(v) = [w_1 + v], \quad v \in S^q. \tag{76}$$

Similarly $E^\perp \cap Q^{n+1}$ is the quadric given in homogeneous coordinates by

$$x_{n+3}^2 = x_{q+3}^2 + \cdots + x_{n+2}^2, \quad x_1 = \cdots = x_{q+2} = 0. \tag{77}$$

This space $E^\perp \cap Q^{n+1}$ is diffeomorphic to the unit sphere S^p in the span $\mathbf{R}^{p+1}$ of the spacelike vectors $w_{q+3}, \ldots, w_{n+2}$ with the diffeomorphism

$$\delta : S^p \to E^\perp \cap Q^{n+1}$$

given by

$$\delta(u) = [u + w_{n+3}], \quad u \in S^p. \tag{78}$$

The image of the curvature sphere map k_1 of multiplicity p is contained in the q-dimensional quadric $E \cap Q^{n+1}$ given by equation (75), which is diffeomorphic to S^q. The map k_1 is constant on each leaf of its principal foliation T_1, and so k_1 factors through an immersion of the q-dimensional space of leaves W/T_1 into the q-dimensional quadric $E \cap Q^{n+1}$. Hence, the image of k_1 is an open subset of this quadric, and each leaf of T_1 corresponds to a point $\gamma(v)$ of the quadric.

Similarly, the curvature sphere map k_2 of multiplicity q factors through an immersion of its p-dimensional space of leaves W/T_2 onto an open subset of the p-dimensional quadric $E^\perp \cap Q^{n+1}$ given by equation (77), and each leaf of T_2 corresponds to a point of $\delta(u)$ of that quadric.

From this it is clear that the restriction of the Legendre map λ to the neighborhood $W \subset M$ is contained in the compact, connected cyclide

$$\nu : S^p \times S^q \to \Lambda^{2n-1}$$

defined by

$$\nu(u,v) = [k_1(u,v), k_2(u,v)], \quad (u,v) \in S^p \times S^q, \tag{79}$$

where

$$k_1(u,v) = \gamma(v), \quad k_2(u,v) = \delta(u), \tag{80}$$

for the maps γ and δ defined above. By a standard connectedness argument, the Legendre map $\lambda : M \to \Lambda^{2n-1}$ is also the restriction of the compact, connected cyclide ν to an open subset of $S^p \times S^q$. This proves part (a) of the theorem.

In projective space P^{n+2}, the image of ν consists of all lines joining a point on the quadric $E \cap Q^{n+1}$ in equation (75) to a point on the quadric $E^\perp \cap Q^{n+1}$ in equation (77). Thus any choice of a $(q+1)$-plane E in P^{n+2} with signature $(q+1,1)$ and corresponding orthogonal complement $E^\perp$ with signature $(p+1,1)$ determines a unique compact, connected cyclide of characteristic (p,q) and vice-versa.

The Lie equivalence of any two compact, connected cyclides of the same characteristic stated in part (b) of the theorem is then clear, since given any two choices E and F of $(q+1)$-planes in P^{n+2} with signature $(q+1,1)$, there is a Lie sphere transformation that maps E to F and $E^\perp$ to $F^\perp$.

In particular, if we take F to be the space Ω in equation (67), then the corresponding cyclide is our standard model. So our given compact, connected cyclide ν in equation (79) is Lie equivalent to our standard model. As noted before the statement of Theorem 10.1, our standard model is Lie equivalent to the Legendre lift of the standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \tag{81}$$

where $p + q = n - 1$, via parallel transformation. Thus, any compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres given in equation (81). $\square$

Remark 10.1. *We can also see that the the submanifold λ in Theorem 10.1 is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n with two principal curvatures by invoking Theorem 9.3, because the two curvature sphere maps $[k_1]$ and $[k_2]$ are orthogonal to the two points w_{n+3} and w_1, respectively, on the timelike line $[w_1, w_{n+3}]$ in P^{n+2}.*

Theorem 10.1 is a classification of the cyclides of Dupin in the context of Lie sphere geometry. It is also useful to have a Möbius geometric classification of the cyclides of Dupin $M^{n-1} \subset \mathbf{R}^n$. This is analogous to the classical characterizations of the cyclides of Dupin in $\mathbf{R}^3$ obtained in the nineteenth century (see, for example, [18, pp. 151–166]). K. Voss [46] announced the classification in Theorem 10.2 below for the higher-dimensional cyclides, but he did not publish a proof. The theorem follows quite directly from Theorem 10.1 and known results on surfaces of revolution.

The theorem is phrased in terms embedded hypersurfaces in $\mathbf{R}^n$. Thus we are excluding the standard model given in equation (69), where the spherical projection (and thus the Euclidean projection) is not an immersion. Of course, the spherical projections of all parallel submanifolds of the standard model in the sphere are embedded isoparametric hypersurfaces in the sphere S^n, except for the Legendre lift of the other focal submanifold. The following proof was first given in [8], and later versions of the proof together with computer graphic illustrations of the cyclides are given in [9, pp. 151–159] and [19, pp. 273–281].

Theorem 10.2. (a) *Every connected cyclide of Dupin $M^{n-1} \subset \mathbf{R}^n$ of characteristic (p, q) is Möbius equivalent to an open subset of a hypersurface of revolution obtained by revolving a q-sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis*

$\mathbf{R}^q \subset \mathbf{R}^{q+1}$ *or a p-sphere* $S^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$ *about an axis* $\mathbf{R}^p \subset \mathbf{R}^{p+1}$.

(b) *Two hypersurfaces obtained by revolving a q-sphere* $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ *about an axis of revolution* $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ *are Möbius equivalent if and only if they have the same value of* $\rho = |r|/a$*, where* r *is the signed radius of the profile sphere* S^q *and* $a > 0$ *is the distance from the center of* S^q *to the axis of revolution.*

Remark 10.2. *Note that in this theorem, the profile sphere* $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ *is allowed to intersect the axis of revolution* $\mathbf{R}^q \subset \mathbf{R}^{q+1}$*, in which case the hypersurface of revolution has singularities in* $\mathbf{R}^n$*. Under Möbius transformation, this leads to cyclides which have Euclidean singularities, such as the classical horn cyclides and spindle cyclides, as discussed in the proof below (see, for example, [9, pp. 151–159] for more detail). In these cases, however, the corresponding Legendre map* $\lambda : S^p \times S^q \to \Lambda^{2n-1}$ *is an immersion.*

Proof. We always work with the Legendre lift of an embedded hypersurface M^{n-1} in $\mathbf{R}^n$. By part (a) of Theorem 10.1, in the context of Lie sphere geometry, every connected cyclide is contained in a unique compact, connected cyclide. Thus, it suffices to classify compact, connected cyclides up to Möbius equivalence. Consider a compact, connected cyclide

$$\lambda : S^p \times S^q \to \Lambda^{2n-1}, \quad p + q = n - 1,$$

of characteristic (p, q). By the proof of Theorem 10.1, there is a linear space E of $\mathbf{P}^{n+2}$ with signature $(q+1, 1)$ such that the two curvature sphere maps,

$$[k_1] : S^q \to E \cap Q^{n+1}, \quad [k_2] : S^p \to E^\perp \cap Q^{n+1},$$

are diffeomorphisms.

Möbius transformations are precisely those Lie sphere transformations A satisfying $A[e_{n+3}] = [e_{n+3}]$. Thus we decompose e_{n+3} as

$$e_{n+3} = \alpha + \beta, \quad \alpha \in E, \quad \beta \in E^\perp. \tag{82}$$

Various cases arise depending on the nature of the two vectors α and β in this decomposition, and this is the key to the proof.

Note first that since $\langle \alpha, \beta \rangle = 0$, we have

$$-1 = \langle e_{n+3}, e_{n+3} \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle.$$

Hence, at least one of the two vectors α, β is timelike. First, suppose that β is timelike. Let Z be the orthogonal complement of β in $E^\perp$. Then Z is a $(p+1)$-dimensional vector space on which the restriction of $\langle\ ,\ \rangle$ has signature $(p+1, 0)$. Since every vector in Z is orthogonal to both α and β, we have $Z \subset e_{n+3}^\perp$ by equation (82). Thus, there is a Möbius transformation A such that

$$A(Z) = S = \text{Span } \{e_{q+3}, \ldots, e_{n+2}\}.$$

The curvature sphere map $[Ak_1]$ of the Dupin submanifold $A\lambda$ is a q-dimensional submanifold in the space $S^\perp \cap Q^{n+1}$. By equation (21), this means that these spheres all have their centers in the space

$$\mathbf{R}^q = \text{Span } \{e_3, \ldots, e_{q+2}\}.$$

Note that

$$\mathbf{R}^q \subset \mathbf{R}^{q+1} = \text{Span } \{e_3, \ldots, e_{q+3}\} \subset \mathbf{R}^n = \text{Span } \{e_3, \ldots, e_{n+2}\}.$$

By Theorem 5.11 of the book [9, pp. 142–143], this means that the Dupin submanifold $A\lambda$ is a hypersurface of revolution in $\mathbf{R}^n$ obtained by revolving a q-dimensional profile submanifold in $\mathbf{R}^{q+1}$ about the axis $\mathbf{R}^q \subset \mathbf{R}^{q+1}$. Moreover, since $A\lambda$ has two distinct curvature spheres, the profile submanifold has only one curvature sphere. Thus, it is an umbilical submanifold of $\mathbf{R}^{q+1}$.

We now have four cases to consider that are naturally distinguished by the nature of the vector α in equation (82). Geometrically, these correspond to different singularity sets of the Euclidean projection of $A\lambda$. Such singularities correspond exactly with the singularities of the Euclidean projection of λ into $\mathbf{R}^n$, since the Möbius transformation A preserves the rank of the Euclidean projection. Since we have assumed that β is timelike, we know that for all $u \in S^p$,

$$\langle k_2(u), e_{n+3}\rangle = \langle k_2(u), \alpha + \beta\rangle = \langle k_2(u), \beta\rangle \neq 0,$$

because the orthogonal complement of β in $E^\perp$ is spacelike. Thus, $[k_2]$ is never a point sphere, and so the curvature sphere $[Ak_2]$ is never a point sphere, since A is a Möbius transformation. However, it is possible for $[Ak_1]$ to be a point sphere.

Case 1: $\alpha = 0$. In this case, the curvature sphere $[Ak_1]$ is a point sphere for every point in $S^p \times S^q$. The image of the Euclidean projection of $A\lambda$ is precisely the axis $\mathbf{R}^q$. The cyclide $A\lambda$ is the Legendre lift induced from $\mathbf{R}^q$

as a submanifold of codimension $p+1$ in $\mathbf{R}^n$. This is, in fact, the standard model given in equation (69). However, since the Euclidean projection is not an immersion, this case does not lead to any of the embedded hypersurfaces classified in part (a) of the theorem.

In the remaining cases, we can always arrange that the umbilic profile submanifold is a q-sphere and not a q-plane. This can be accomplished by first inverting $\mathbf{R}^{q+1}$ in a sphere centered at a point on the axis $\mathbf{R}^q$ which is not on the profile submanifold, if necessary. Such an inversion preserves the axis of revolution $\mathbf{R}^q$.

After a Euclidean translation, we may assume that the center of the profile sphere is a point $(0, a)$ on the x_{q+3}-axis ℓ in $\mathbf{R}^{q+1}$. The center of the profile sphere cannot lie on the axis of revolution $\mathbf{R}^q$, for then the hypersurface of revolution would be an $(n-1)$-sphere and not a cyclide of Dupin. Thus, we may take $a > 0$.

The map $[Ak_1]$ is the curvature sphere map that results from the surface of revolution construction. The other curvature sphere of $A\lambda$ corresponds exactly to the curvature sphere of the profile sphere, i.e., to the profile sphere itself. This means that the signed radius r of the profile sphere is equal to the signed radius of the curvature sphere $[Ak_2]$. Since $[Ak_2]$ is never a point sphere, we conclude that $r \neq 0$. From now on, we will identify the profile sphere with the second factor S^q in the domain of λ.

Case 2: *α is timelike.* In this case, for all $v \in S^q$, we have

$$\langle k_1(v), e_{n+3} \rangle = \langle k_1(v), \alpha \rangle \neq 0,$$

since the orthogonal complement of α in E is spacelike. Thus the Euclidean projection of $A\lambda$ is an immersion at all points. This corresponds to the case $|r| < a$, when the profile sphere is disjoint from the axis of revolution. Note that by interchanging the roles of α and β, we can find a Möbius transformation that takes λ to the Legendre submanifold obtained by revolving a p-sphere around an axis $\mathbf{R}^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$.

In the classical situation of surfaces in $\mathbf{R}^3$, the Euclidean projection of $A\lambda$ above is a torus of revolution. Its focal set consists of the core circle and the axis of revolution covered twice. This is a special case of a pair of focal conics. The Euclidean projection of λ itself is a *ring cyclide* if the Möbius projection of λ does not contain the improper point, or an unbounded *parabolic ring cyclide* if the Möbius projection of λ does contain the improper point.

In the classical situation, the focal set in $\mathbf{R}^3$ consists of a pair of *focal conics* for both types of surfaces. Specifically, for a ring cyclide, the focal set consists of an ellipse and hyperbola in mutually orthogonal planes such that the vertices of the ellipse are the foci of the hyperbola and vice-versa. For a parabolic ring cyclide, the focal set consists of a pair of parabolas in orthogonal planes such that the vertex of each is the focus of the other. (See [9, pp. 151–159] for computer graphics illustrations of these cyclides.)

Case 3: *α is lightlike, but not zero.* Then there is exactly one $v \in S^q$ such that

$$\langle k_1(v), e_{n+3} \rangle = \langle k_1(v), \alpha \rangle = 0. \tag{83}$$

This corresponds to the case $|r| = a$, where the profile sphere intersects the axis in one point. Thus, $S^p \times \{v\}$ is the set of points in $S^p \times S^q$ where the Euclidean projection is singular.

In the classical situation of surfaces in $\mathbf{R}^3$, the Euclidean projection of $A\lambda$ is a *limit torus*, and the Euclidean projection of λ itself is a *limit spindle cyclide* or a *limit horn cyclide*, if the Möbius projection of λ does not contain the improper point. (See [9, pp. 151–159] for computer graphics illustrations of these cyclides.)

In the case where the Möbius projection of λ contains the improper point, the Euclidean projection of λ is either a *limit parabolic horn cyclide* or a circular cylinder (in the case where the singularity is at the improper point). For all of these surfaces except the cylinder, the focal set consists of a pair of focal conics, as in the previous case. For the cylinder, the Euclidean focal set consists only of the axis of revolution, since one of the principal curvatures is identically zero, and so the corresponding focal points are all at infinity. In Lie sphere geometry, both curvature sphere maps are plane curves on the Lie quadric, as shown in the proof of Theorem 10.1.

Case 4: *α is spacelike.* Then the condition (83) holds for points v in a $(q-1)$-sphere $S^{q-1} \subset S^q$. For points in $S^p \times S^{q-1}$, the point sphere map is a curvature sphere, and thus the Euclidean projection is singular. Geometrically, this is the case $|r| > a$, where the profile sphere intersects the axis $\mathbf{R}^q$ in a $(q-1)$-sphere.

In the classical situation of surfaces in $\mathbf{R}^3$, the Euclidean projection of $A\lambda$ is a *spindle torus*, and the Euclidean projection of λ itself is a *spindle cyclide* or a *horn cyclide*, if the Möbius projection does not contain the improper point.

In the case where the Möbius projection of λ contains the improper point, the Euclidean projection of λ is either a *parabolic horn cyclide* or circular cone (in the case where one of the singularities is at the improper point). For all of these surfaces except the cone, the focal set consists of a pair of focal conics. For the cone, the Euclidean focal set consists of only the axis of revolution (minus the origin), since one principal curvature is identically zero. (See [9, pp. 151–159] for computer graphics illustrations of these cyclides.)

Of course, there are also four cases to handle under the assumption that α, instead of β, is timelike. Then the axis will be a subspace $\mathbf{R}^p \subset \mathbf{R}^{p+1}$, and the profile submanifold will be a p-sphere. The roles of p and q in determining the dimension of the singularity set of the Euclidean projection will be reversed. So if $p \neq q$, then only a ring cyclide can be represented as a hypersurface of revolution of both a q-sphere and a p-sphere. This completes the proof of part (a).

To prove part (b), we may assume that the profile sphere S^q of the hypersurface of revolution has center $(0, a)$ with $a > 0$ on the x_{q+3}-axis ℓ. Möbius classification clearly does not depend on the sign of the radius of S^q, since the two hypersurfaces of revolution obtained by revolving spheres with the same center and opposite radii differ only by the change of orientation transformation mentioned in Section 5.

We now show that the ratio $\rho = |r|/a$ is invariant under the subgroup of Möbius transformations of the profile space $\mathbf{R}^{q+1}$ which take one such hypersurface of revolution to another. First, note that symmetry implies that a transformation T in this subgroup must take the axis of revolution $\mathbf{R}^q$ to itself and the axis of symmetry ℓ to itself. Since $\mathbf{R}^q$ and ℓ intersect only at 0 and the improper point ∞, the transformation T maps the set $\{0, \infty\}$ to itself. If T maps 0 to ∞, then the composition ΦT, where Φ is an inversion in a sphere centered at 0, is a member of the subgroup of transformations that map ∞ to ∞ and map 0 to 0.

By Theorem 3.16 of [9, p. 47], such a Möbius transformation must be a similarity transformation, and so it is the composition of a central dilatation D and a linear isometry Ψ. Therefore, $T = \Phi D \Psi$, and each of the transformations on the right side of this equation preserves the ratio ρ. The invariant ρ is the only one needed for Möbius classification, since any two profile spheres with the same value of ρ can be mapped to one another by a central dilatation. □

Remark 10.3. *We can obtain a family consisting of one representative from each Möbius equivalence class by fixing* $a = 1$ *and letting* r *vary,* $0 < r < \infty$*. This is just a family of parallel hypersurfaces of revolution. Taking a negative signed radius* s *for the profile sphere yields a parallel hypersurface that differs only in orientation from the hypersurface corresponding to* $r = -s$*. Finally, taking* $r = 0$ *also gives a parallel submanifold in the family, but the Euclidean projection degenerates to a sphere* S^p*. This is the case* $\beta = 0, \alpha = e_{n+3}$*, where the point sphere map equals the curvature sphere* $[k_2]$ *at every point.*

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