2-27-2023

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Translator's Note: This is an unofficial translation of the original paper which was written in French. All references should be made to the original paper.

Mathematische Zeitschrift **45**, *335–367 (1939).*

On remarkable families of isoparametric hypersurfaces in spherical spaces.

by

Élie Cartan (in Paris).

In a recent article[1] I showed the existence of a family of isoparametric hypersurfaces in the spherical space of four dimensions with three distinct principal curvatures. I propose to study whether there exist, in a spherical space of any number of dimensions, families of isoparametric hypersurfaces admitting exactly three distinct principal curvatures. In the first part of this Memoir, I will show the existence of such families in the spaces of 4, 7, 13, and 25 dimensions; this existence is linked to that of a homogeneous polynomial of third degree with $n+2$ variables ($n+1$ being the dimension of the spherical space) enjoying the double property that its first differential

[1] Annali di Mat. **17** (1938), p. 177–191.

parameter, calculated in the euclidean space of $n+2$ dimensions with rectangular coordinates $x_1, \ldots, x_{n+2}$, is constant on the hypersphere of radius 1 and its second differential parameter is zero (harmonic polynomial). Such polynomials exist only for $n = 3,\ 6,\ 12,\ 24$ or $n = 3 \cdot 2^k$ $(k = 0, 1, 2, 3)$. The case $n = 24$ is particularly interesting because it is linked to different theories (theory of spinors, principle of triality in the elliptical space of 7 dimensions) and that it provides the first appearance in a problem of Geometry (and also of Analysis) of the simple group with 52 parameters that does not fit into any of the major classes of simple groups.

In the second part of the Memoir, I will show that the families of isoparametric hypersurfaces determined in this way are the only ones for which the hypersurfaces have exactly three distinct principal curvatures. It is only in this second part that an appeal will be made to the results and formulas of the Memoir recalled in the footnote. Finally, I will prove a very simple fundamental theorem on families of isoparametric hypersurfaces whose principal curvatures are all of the same degree of multiplicity.

First Part.

I. Determination of a class of polynomials.

1. We consider in euclidean space of $n+2$ dimensions with respect to rectangular coordinates $x_1, x_2, \ldots, x_{n+2}$ an entire polynomial $F(x)$ of the third degree. On the hypersphere of radius 1 centered at the origin, this polynomial, if it is not identically zero, has an absolute maximum and an absolute minimum equal to and opposite to the maximum. We will assume that the maximum is equal to 1. We will assume further

1° that the first differential parameter

$$\Delta_1 F = \sum_i \left(\frac{\partial F}{\partial x_i}\right)^2$$

has a constant value on the hypersphere and therefore satisfies a relation of the form

$$\Delta_1 F \equiv \sum_i \left(\frac{\partial F}{\partial x_i}\right)^2 = \lambda(x_1^2 + x_2^2 + \cdots + x_{n+2}^2)^2, \tag{1}$$

λ being a constant;

2° that the polynomial F is harmonic:

$$\Delta_2 F \equiv \sum_i \frac{\partial^2 F}{\partial x_i^2} = 0. \tag{2}$$

We will get the value of λ in the following way. If the polynomial F takes its maximum value 1 at the point (x_i), we will have at this point

$$\frac{\partial F}{\partial x_i} = \rho x_i,$$

whence, by multipication by x_i and summation with respect to i,

$$3F = \rho, \quad \text{or} \quad \rho = 3;$$

squaring and summing, we get $\lambda = \rho^2 = 9$. We can therefore write

$$\sum_i \left(\frac{\partial F}{\partial x_i}\right)^2 = 9\ (x_1^2 + x_2^2 + \cdots + x_{n+2}^2)^2. \tag{1'}$$

Before solving equations (1′) and (2) we will assume, without loss of generality, that the polynomial F takes its maximum value 1 at the point $x_1 = \cdots = x_{n+1} = 0$, $x_{n+2} = 1$ of the hypersphere of radius 1; we will therefore have at this point

$$\frac{\partial F}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n+1).$$

We can then write

$$F \equiv x_{n+2}^3 + x_{n+2}\ P(x) + Q(x), \tag{3}$$

the polynomials P and Q only depend on $x_1, x_2, \ldots, x_{n+1}$.

The property that the polynomial F is harmonic gives the immediate relations

$$\Delta_2 P + 6 = 0, \tag{4}$$

$$\Delta_2 Q = 0. \tag{5}$$

As for the identity (1′), it gives the three relations

$$6P + \sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 = 18 \sum x_i^2, \tag{6}$$

$$\sum_i \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_i} = 0, \tag{7}$$

$$P^2 + \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 = 9\ (x_1 + x_2 + \cdots + x_{n+1}^2)^2\ ; \tag{8}$$

the summation in these formulas is made with respect to the indices $1, 2, \ldots,$ $n+1$.

We are going to solve equations (4), (5), (6), (7), (8).

2. Equation (4) is easy to solve. P being a polynomial of second degree is, by a suitable choice of the coordinate axes, reducible to

$$P \equiv a_1\ x_1^2 + a_2\ x_2^2 + \cdots + a_{n+1}\ x_{n+1}^2\ ;$$

the relation (4) gives

$$a_1 + a_2 + \cdots + a_{n+1} = -3.$$

If we now put the value of P into the relation (6), we obtain for each value of i,

$$4a_i^2 + 6a_i - 18 = 0,$$

so that each of the coefficients of the polynomial has one of the values $3/2$ and -3. If the first occurs p times and the second q times ($p + q = n + 1$), we have

$$a_1 + a_2 + \cdots + a_{n+1} = \frac{3}{2}\ (p - 2q),$$

whence

$$p - 2q = -2, \quad p = 2\ (q - 1).$$

We set

$$q = \nu + 1, \quad p = 2\nu, \quad \text{whence}\ \ n = 3\nu,$$

and we will have

$$P \equiv \frac{3}{2}(x_1^2 + x_2^2 + \cdots + x_{2\nu}^2) - 3\ (x_{2\nu+1}^2 + \cdots + x_{3\nu+1}^2). \tag{9}$$

3. We now proceed to the determination of the polynomial Q. But first we will change notations. We will denote by $z_\alpha (\alpha = 1, 2, \ldots, \nu+1)$ the coordinates $x_{2\nu+1}, \ldots, x_{3\nu+1}$; by x_i $(i = 1, 2, \ldots, 2\nu)$ the coordinates $x_1, x_2, \ldots, x_{2\nu}$, and by u the coordinate x_{n+2}, so that we have

$$P \equiv \frac{3}{2} \sum_i x_i^2 \; - \; 3 \sum_\alpha z_\alpha^2. \tag{9'}$$

The polynomial of third degree Q can be decomposed into four parts

$$Q = A + B + C + D,$$

A being homogeneous of degree 3 with respect to the x_i, B homogeneous of degree 2 with respect to the x_i and degree 1 respect to the z_α and so on.

Given this, the relation (7) gives immediately

$$9A - 9C - 18D = 0,$$

whence $A = C = D = 0$ and then we can write

$$Q = \sum_\alpha z_\alpha Q_\alpha(x), \tag{10}$$

the Q_α being polynomials of the second degree in $x_1, x_2, \ldots, x_{2\nu}$. Relation (5) then shows that these polynomials are harmonic. Finally the relation (8) gives

$$\frac{9}{4}\left(\sum x_i^2\right)^2 - 9\sum x_i^2 \sum z_\alpha^2 + 9\left(\sum z_\alpha^2\right)^2 + \sum_\alpha [Q_\alpha(x)]^2 + \sum_i \left(\sum_\alpha z_\alpha \frac{\partial Q_\alpha}{\partial x_i}\right)^2$$
$$= 9\left(\sum x_i^2\right)^2 + 18\sum x_i^2 \sum z_\alpha^2 + 9\left(\sum z_\alpha^2\right)^2;$$

it decomposes into

$$\sum_\alpha [Q_\alpha(x)]^2 = \frac{27}{4}\left(\sum x_i^2\right)^2, \tag{8'}$$

$$\sum_i \left(\sum_\alpha z_\alpha \frac{\partial Q_\alpha}{\partial x_i}\right)^2 = \sum x_i^2 \sum z_\alpha^2. \tag{8''}$$

Let us first take the relation (8″); it gives

$$\sum_i \frac{\partial Q_\alpha}{\partial x_i}\,\frac{\partial Q_\beta}{\partial x_i} = 27\ \delta_\alpha^\beta \sum x_i^2 \quad (\delta_\alpha^\beta = 1 \text{ if } \alpha = \beta,\ \delta_\alpha^\beta = 0 \text{ if } \alpha \neq \beta);$$

in particular the harmonic polynomial $Q_{\nu+1}(x)$ enjoys the property (like the others) that its first differential parameter is equal to $27 \sum x_i^2$; we can reduce it to the form

$$Q_{\nu+1} \equiv b_1 x_1^2 + b_2 x_2^2 + \cdots + b_{2\nu} x_{2\nu}^2,$$

with

$$b_1^2 = b_2^2 = \cdots = b_{2\nu}^2 = \frac{27}{4}, \quad b_1 + b_2 + \cdots + b_{2\nu} = 0;$$

as a result, we can assume

$$Q_{\nu+1} = \frac{3\sqrt{3}}{2}(x_1^2 + \cdots + x_\nu^2 - x_{\nu+1}^2 - \cdots - x_{2\nu}^2).$$

We will introduce yet another new notation; we will reserve the letter x for the first ν coordinates $x_1, x_2, \ldots, x_\nu$ and we will use the letter y for the ν following ones, which we will call $y_1, y_2, \ldots, y_\nu$; the letter i will now be used for the indices $1, 2, \ldots, \nu$. Finally, we will use the letter v for the coordinate $z_{\nu+1}$. We can therefore write

$$Q_{\nu+1} \equiv \frac{3\sqrt{3}}{2}\left(\sum x_i^2 - \sum y_i^2\right). \tag{11}$$

The relations (8) now take the form

$$\begin{cases} \sum_i [Q_i(x,y)]^2 = 27 \sum x_i^2 \sum y_i^2, \\ \sum_i \left(x_i \frac{\partial Q_k}{\partial x_i} - y_i \frac{\partial Q_k}{\partial y_i}\right) = 0 \quad (k = 1, 2, \ldots, \nu), \\ \sum_i \left[\left(\frac{\partial Q_k}{\partial x_i}\right)^2 + \left(\frac{\partial Q_k}{\partial y_i}\right)^2\right]^2 = 27 \left(\sum x_i^2 + \sum y_i^2\right), \\ \sum_i \left(\frac{\partial Q_k}{\partial x_i}\frac{\partial Q_h}{\partial x_i} + \frac{\partial Q_k}{\partial y_i}\frac{\partial Q_h}{\partial y_i}\right) = 0 \quad (k \neq h). \end{cases} \tag{12}$$

The second relation (12) shows that the polynomials $Q_k(x,y)$ are bilinear with respect to the two series of variables x_i, y_i:

$$Q_k(x) = \sum_{i,j} a_{ijk} x_i y_j.$$

4. If we summarize the results obtained, we see that the polynomial F takes the form

$$\begin{aligned} F \equiv u^3 - 3uv^2 \quad &+ \quad \frac{3}{2}u \sum_i (x_i^2 + y_i^2) - 3u \sum_i z_i^2 \\ &+ \quad \frac{3\sqrt{3}}{2}v \sum (x_i^2 - y_i^2) + \sum_k z_k \; Q_k(x, y). \end{aligned} \tag{13}$$

The trilinear form $\sum z_k Q_k$ of the three series of variables x_i, y_i, z_i enjoys some remarkable properties. If we set

$$Q_k = 3\sqrt{3}\; H_k,$$

we see that the first relation (12) provides a generalization of the well-known formulas of Lagrange and of Brioschi which represent the product of two sums of ν squares by a sum of ν squares. We have in fact

$$\sum_i [H_i(x, y)]^2 = \sum x_i^2 \sum y_i^2, \tag{14}$$

the ν quantities $H_i(x, y)$ of the first member being bilinear with respect to the x_i and the y_i. This relationship leads to all of the other relations (12), for if we set

$$H_i(x, y) = \sum_k a_{ik}(x) y_k,$$

the relation (14) shows that in the table with ν rows and ν columns of coefficients $a_{ik}(x)$, the sum of the squares of the elements of the same column is equal to $\sum x_i^2$ and the sum of the products of the elements of two different columns taken successively in the ν rows is zero; this is precisely what the relations in (12) other than the first express. We also see at the same time that the deteminant of this array is not zero and is equal to $[\sum x_i^2]^{\frac{\nu}{2}}$.

But the relation (14) has a remarkable geometric significance. To highlight this, it will be convenient to consider a particular one of the polynomials H_i, for example H_1. We have

$$H_1 = a_1(x)y_1 + a_2(x)y_2 + \cdots + a_\nu(x)y_\nu$$

with

$$a_1^2 + a_2^2 + \cdots + a_\nu^2 = x_1^2 + x_2^2 + \cdots + x_\nu^2$$

consequently we can suppose, by a suitable linear orthogonal substitution carried out on the x_i, that $a_i(x) = x_i$, whence

$$H_1 = x_1 y_1 + x_2 y_2 + \cdots + x_\nu y_\nu. \tag{15}$$

That being said, we will regard the x_i and the y_i as homogeneous coordinates on the elliptical space of $\nu - 1$ dimensions; they will be normal coordinates if the sum of their squares is equal to 1. If the form $H_1(x, y)$ vanishes, this means that the two points are conjugate to each other with respect to the absolute. They define a line. We also know that if (y_i) and (z_i) are both conjugates of (x_i) with respect to the absolute, and if the coordinates of the three points (x), (y), (z) are normal, the quantity $\sum_i y_i z_i$ represents the cosine of the angle of the two lines $[xy]$ and $[xz]$.

5. We now come to the announced geometric interpretation. We will agree to say that two lines $[xy]$ and $[x'y']$, y being a conjugate of x and y' of x', are *parallel* if, the coordinates x_i, y_i, x'_i, y'_i being normal, we have

$$H_i(x, y) = H_i(x', y').$$

Through any point x' there passes a well-determined parallel to the line $[xy]$, because the determinant of the coefficients of y'_i in $H_i(x', y')$ is equal to 1. Two lines parallel to a third line are parallel to each other. *Therefore, there exists an absolute parallelism in the elliptic space of $\nu - 1$ dimensions.* Moreover, this *absolute parallelism is isogonal*, that is to say, that the angle of intersection of two straight lines emanating from a common point is equal to the angle of intersection of the parallels to these two straight lines passing through a common point. Indeed, let $[x'y']$ and $[x'z']$ be parallel to $[xy]$ and $[xz]$; from the equalities

$$H_i(x', y') = H_i(x, y),$$
$$H_i(x', z') = H_i(x, z),$$

results the equality

$$H_i(x', y' + \lambda z') = H_i(x, y + \lambda z);$$

by squaring, adding and taking into account (14) and the assumption that the coordinates introduced are all normal, we immediately find

$$1 + \lambda^2 + 2\lambda \sum y'_i z'_i = 1 + \lambda^2 + 2\lambda \sum y_i z_i,$$

that is,

$$\sum y_i' z_i' = \sum y_i z_i. \qquad \text{C.Q.F.D.}$$

6. It is known[2] that the only riemannian spaces admitting an isogonal absolute parallelism are the representative spaces of closed simple groups and the elliptical space of 7 dimensions, to which must be added the representative space of the closed group of rotations of the circumference ($\nu = 2$), as well as the topological products of two or more of the preceding spaces. Among all these spaces, those which are of constant curvature are the elliptical spaces of 1, 3, and 7 dimensions; no topological product fits, because the ds^2 of the corresponding riemannian space would be the sum of two ds^2 depending on the separate variables u_i and v_j, and the Riemann tensor could not be of the form that fits a space with nonzero constant curvature.

Consequently, the proposed problem only admits a solution for $\nu = 2, 4, 8$, cases to which it is naturally necessary to add $\nu = 1$!

It follows from the well-known theory of isogonal parallelisms in the cases which have just been demonstrated that the only possibilities that one can give to the trilinear form $\mathfrak{F} = \sum z_k H_k(x, y)$ have the general expression

$$\mathfrak{F} = \frac{1}{2}(XYZ + \overline{Z}\ \overline{Y}\ \overline{X}); \tag{16}$$

for $\nu = 1$, X, Y, Z are three real variables and

$$\overline{X} = X, \quad \overline{Y} = Y, \quad \overline{Z} = Z;$$

for $\nu = 2$, X, Y, Z are three complex variables, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are their three conjugate variables;

for $\nu = 4$, X, Y, Z are three quaternions, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are their conjugates;

for $\nu = 8$, X, Y, Z are three octaves of Graves-Cayley, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are their conjugates; in this last case, the multiplication of octaves not being associative, it is necessary to interpret XYZ as being $(XY)Z$, and $\overline{Z}\ \overline{Y}\ \overline{X}$ as the conjugate octave $\overline{Z}\ (\overline{Y}\ \overline{X})$.

Finally, the formula (13) becomes

$$\begin{aligned} F \equiv u^3 - 3uv^2 \ &+ \ \frac{3}{2}\ u\ (X\overline{X} + Y\overline{Y} - 2\ Z\overline{Z}) \\ &+ \ \frac{3\sqrt{3}}{2}\ v\ (X\overline{X} - Y\overline{Y}) + \frac{3\sqrt{3}}{2}(XYZ + \overline{Z}\ \overline{Y}\ \overline{X}). \end{aligned} \tag{17}$$

[2] E. Cartan and J. A. Schouten, *On the riemannian Geometries admitting an absolute parallelism* [Proc. Akad. Amsterdam **29** (1926), p. 933–946].

II. Properties of hypersurfaces $F = C$.

7. We will first verify that the hypersurfaces $F = C$ of the spherical space of $3\nu + 1$ dimensions form a family of parallel hypersurfaces. Indeed, let us return to our initial notations; let x be a point on the hypersurface $F = C$; an infinitely close point $x + \delta x$ is in the direction normal to this hypersurface if, the coordinates x_i being assumed to be normal, the point δx is conjugate with respect to the points $x + dx$ infinitely close to x on the hypersurface, in other words if the relation $\sum \delta x_i dx_i = 0$ is a consequence of the two relations

$$\sum_i \frac{\partial F}{\partial x_i} dx_i = 0, \quad \sum_i x_i dx_i = 0;$$

so we have

$$\delta x_i = \lambda \frac{\partial F}{\partial x_i} + \mu x_i.$$

The point δx must also be conjugate with respect to the point x, which gives

$$3\lambda C + \mu = 0,$$

whence

$$\delta x_i = \lambda \left(\frac{\partial F}{\partial x_i} - 3C x_i \right).$$

To get λ, let us say that the point $x + \delta x$ belongs to the hypersurface with parameter $C + \delta C$, which gives

$$\delta C = \lambda\ (9 - 9C^2) = 9\lambda\ (1 - C^2).$$

Finally, by calling δt the distance between the two points x and $x + \delta x$, we find

$$\delta t = \lambda \sqrt{9(1 - C^2)} = \frac{1}{3} \frac{\delta C}{\sqrt{1 - C^2}}.$$

This formula proves that the hypersurfaces form a family of parallel hypersurfaces; we have on the other hand, t being defined only up to a constant, $C = \cos 3t$, whence

$$\cos 3t = F$$

for the equation of the family. This result could have been easily derived from the considerations of the previously mentioned article.

8. To study the properties of the hypersurfaces of the family, we will first take the case $t = 0$: we have the locus of points where the function F takes its maximum value 1. It is a singular variety of the family, which we have seen (n°1), satisfies the equations

$$\frac{\partial F}{\partial x_i} = 3x_i. \tag{18}$$

Note first that with the polynomial F having constant coefficients and having been constructed starting from any point

$$A \ (x_{n+2} = 1, x_1 = \cdots = x_{n+1} = 0)$$

of the variety under consideration, this variety admits a transitive group of rigid displacements of the ambient space. Let us now recall that the curvature at A, calculated in the spherical space under consideration, realized by the hypersphere of radius 1 of the euclidean space of $n+2$ dimensions, of any curve drawn on the variety and passing through A is conserved by orthogonal projection onto the tangent hyperplane to the hypersphere at A, the hyperplane with equation $x_{n+2} = 1$, or in the new notations, $u = 1$. Given this, the equations (18) of V in a neighborhood of A, limited to terms of the second degree, will be according to (13),

$$3\ u^2 - 3\ v^2 + \frac{3}{2}\ (X\overline{X} + Y\overline{Y} - 2\ Z\overline{Z}) = 3\ u,$$

$$-6\ u\ v + \frac{3\sqrt{3}}{2}\ (X\overline{X} - Y\overline{Y}) = 3\ v,$$

$$3\ u\ x_i + 3\sqrt{3}\ v\ x_i + \sum_k z_k \frac{\partial Q_k}{\partial x_i} = 3\ x_i,$$

$$3\ u\ y_i - 3\sqrt{3}\ v\ y_i + \sum_k z_k \frac{\partial Q_k}{\partial y_i} = 3\ y_i,$$

$$-6\ u\ z_i + Q_i(x,y) = 3\ z_i,$$

or again, by noting that we have, to the same degree of approximation,

$$u = 1 - \frac{1}{2}\sum_i (x_i^2 + y_i^2 + z_i^2) - \frac{1}{2}v^2,$$

and always keeping only the terms of the second degree at most,

$$\begin{cases} z_i = \frac{1}{9} Q_i\,(x,y) = \frac{1}{\sqrt{3}} H_i\,(x,y)\,, \\[2ex] v = \frac{\sqrt{3}}{6} \sum_i (x_i^2 - y_i^2) = \frac{1}{2\sqrt{3}} \sum_i (x_i^2 - y_i^2)\,. \end{cases} \tag{19}$$

This shows that *the variety V has 2ν dimensions*, the plane element tangent at the point A being $dz_i = dv = 0$. The variety of 2ν dimensions defined in a Euclidean space of $3\nu + 1$ dimensions by the equations (19) has the same curvature at A for all curves which pass through this point as the variety V in the spherical space of $3\nu + 1$ dimensions. The normal curvature of any one of these curves is equal to

$$\frac{1}{\sqrt{3}} \frac{\sqrt{[\sum (x_i^2 - y_i^2)]^2 + 4 \sum_i H_i^2}}{\sum (x_i^2 + y_i^2)^2} = \frac{1}{\sqrt{3}};$$

the variety therefore enjoys the property that all the curves drawn on it have the same constant normal curvature $\frac{1}{\sqrt{3}}$.[3] This is a property that had already been verified in the case $\nu = 1$.[4]

9. Let us now study a generic hypersurface Σ of the family, corresponding to the value t of the parameter and take a point M of this hypersurface. The orthogonal trajectory of the hypersurfaces which passes through M will meet the singular variety V at a point P. It is easy to see that the hypersurface Σ admits a transitive group of rigid displacements of the ambient space. Indeed, one can always pass by a displacement of the ambient space from the point P of V to a particular point, that which we designated by A; on the other hand, the equations of the normal plane element at A to V are

$$dx_i = dy_i = 0;$$

now it follows from the way that we have calculated the polynomial F that to any orthogonal substitution on $z_1, z_2, \ldots, z_{\nu+1}$ there corresponds at least one orthogonal substitution on the 2ν coordinates x_i, y_i such that the polynomial

[3]Certain surfaces (of two dimensions) immersed in an elliptical space and enjoying the property that all their curves have the same constant normal curvature have been encountered by O. Boruvka: *Sur les surfaces représentées par les fonctions sphériques de première espèce* [J. d. Math. **12** (1933), p. 337–383].

[4]E. Cartan, loc. cit.[1]), p. 189.

F remains invariant; this amounts to saying that the geodesic of the space which goes from M to P can always be brought back to the geodesic $x_i = y_i = z_i = 0$. In other words, *we can always perform a rigid displacement of the ambient space leaving the hypersurface Σ invariant and transforming the point M to the point with coordinates*

$$x_i = 0, \quad y_i = 0, \quad v = \sin t, \quad u = \cos t.$$

Now we make the orthogonal change of coordinates,

$$u + iv = (U + iV)e^{it},$$

which makes the coordinates of the given point of Σ equal to zero except for $U = 1$. Since we have

$$u^3 - 3\ uv^2 = \frac{1}{2}(u+iv)^3 + \frac{1}{2}(u-iv)^3,$$

the equation of Σ will become

$$\begin{aligned} \cos 3t &= (U^3 - 3\ UV^2)\cos 3t + (V^3 - 3\ U^2V)\sin 3t \\ &+ \frac{3}{2}(U\cos t - V\sin t)\sum(x_i^2 + y_i^2 - 2z_i^2) \\ &+ \frac{3\sqrt{3}}{2}(U\sin t + V\cos t)\sum(x_i^2 - y_i^2) + \sum z_k Q_k(x,y). \end{aligned}$$

Replacing U by $1 - \frac{1}{2}V^2 - \frac{1}{2}\sum(x_i^2 + y_i^2 + z_i^2)$ and neglecting the terms of degree larger than the second, we obtain after simplifications

$$\begin{aligned} V\sin 3t &= \frac{1}{2}(\cos t + \sqrt{3}\sin t - \cos 3t)\sum x_i^2 \\ &+ \frac{1}{2}(\cos t - \sqrt{3}\sin t - \cos 3t)\sum y_i^2 - \frac{1}{2}\,(2\cos t + \cos 3t)\sum z_i^2. \end{aligned} \tag{20}$$

In this form, we see that the hypersurface Σ admits at the point M three distinct and constant principal curvatures, namely

$$\frac{\cos t + \sqrt{3}\sin t - \cos 3t}{\sin 3t}, \quad \frac{\cos t - \sqrt{3}\sin t - \cos 3t}{\sin 3t}, \quad \frac{-2\cos t - \cos 3t}{\sin 3t}.$$

The mean curvature of the hypersurface is thus $-3\nu\cot 3t$. An easy transformation shows, moreover, that the three principal curvatures are

$$\frac{\cot t + \sqrt{3}}{\sqrt{3}\cot t - 1} = -\cot\left(t - \frac{\pi}{3}\right), \ \frac{\cot t - \sqrt{3}}{-\sqrt{3}\cot t - 1} = -\cot\left(t + \frac{\pi}{3}\right), \ -\cot t.$$

We find the expressions predicted by the general theory in the case of three distinct principal curvatures of the same degree of multiplicity (here ν)[5].

10. We have already had the occasion to observe that each hypersurface Σ is invariant under a transitive group of rigid displacements, and it also follows from the reasoning that this group is the same for all hypersurfaces of the family. In reality, there are three distinct groups to consider:

1° the group G of all displacements which leave the polynomial F invariant;

2° the group G_1 of all displacements which leave the various hypersurfaces Σ invariant while leaving fixed a point A of the singular variety V;

3° the group G_2 of all displacements which leave the various hypersurfaces Σ invariant while leaving fixed a point M not located on V;

Each of these groups is a subgroup of the previous one. This is obvious for the second. The third can be regarded as the subgroup of the group which leaves invariant the point A of V where the orthogonal trajectory to the surface Σ starting from M ends, as well as the geodesic AM.

If we know the order r_2 of the third group, then we can easily deduce the orders of the first two. Indeed, the point A of V being fixed, the geodesic AM can take any direction in the plane element of $\nu + 1$ dimensions normal to V at A; we thus have for the order r_1 de G_1 the relation

$$r_1 = r_2 + \nu.$$

On the other hand, the variety V being of 2ν dimensions, and the subgroup of G which leaves fixed a point A of that variety being of r_1 parameters, we have,

$$r = r_1 + 2\nu = r_2 + 3\nu.$$

Now the group G_2 is none other than the group of orthogonal substitutions which leave invariant the form F as well as the coordinates u and v; according

[5]E. Cartan, loc. cit.[1]), p. 187.

to the form of F, it can be characterized as orthogonally transforming each of the series of variables x_i, y_i, z_i and leaving invariant the trilinear form

$$\mathfrak{F} = \sum z_i H_i(x, y).$$

We are going to review the four possible cases, focusing mainly on the last three.

III. The case of the spherical space of 7 dimensions.

11. Recall that in the case $\nu = 1$ (spherical space of 4 dimensions), the group G has three parameters, isomorphic to the group of rotations of ordinary space. The group G_2, which leaves the form $\mathfrak{F} \equiv xyz$ invariant, includes only the 4 operations

$$x' = \pm x, \quad y' = \pm y, \quad z' = \pm z,$$

the product of the three double signs being equal to 1. The group G_1 is the one which leaves invariant the forms

$$x^2 + y^2, \quad z^2 + v^2, \quad (x^2 - y^2)v + 2xyz;$$

it decomposes into the two families

$$\begin{aligned} x' = x\cos\alpha - y\sin\alpha, \quad y' &= x\sin\alpha + y\cos\alpha, \quad z' = z\cos 2\alpha + v\sin 2\alpha, \\ v' &= -z\sin 2\alpha + v\cos 2\alpha; \end{aligned}$$

$$\begin{aligned} x' = x\cos\alpha + y\sin\alpha, \quad y' &= x\sin\alpha - y\cos\alpha, \quad z' = -z\cos 2\alpha + v\sin 2\alpha, \\ v' &= z\sin 2\alpha + v\cos 2\alpha. \end{aligned}$$

Finally, let us recall that the singular surface V can be defined parametrically by the formulas[6]

$$\left\{ \begin{array}{l} x = \sqrt{3}\ \eta\ \zeta, \quad y = \sqrt{3}\ \zeta\ \xi, \quad z = \sqrt{3}\ \xi\ \eta, \\ \\ v = \sqrt{3}\ \frac{\xi^2 - \eta^2}{2}, \quad u = \zeta^2 - \frac{\xi^2 + \eta^2}{2}, \end{array} \right. \tag{21}$$

[6]E. Cartan, loc. cit.[1]), p. 189; the parameters called here ξ, η, ζ were designated by the letters u, v, w.

where ξ, η, ζ are three parameters linked by the relation

$$\xi^2 + \eta^2 + \zeta^2 = 1. \tag{22}$$

The group G is the one induced on the variables x, y, z, u, v by the most general orthogonal linear substitution on ξ, η, ζ.

12. In the case $\nu = 2$, we introduce three complex variables X, Y, Z, into the form F, and the form $\mathfrak{F}$ is none other than the real part of the product XYZ. The group G_2 decomposes into the two continuous families

$$\begin{aligned} X' &= Xe^{i\alpha}, \quad Y' = Ye^{i\beta}, \quad Z' = Ze^{i\gamma} \quad (\alpha + \beta + \gamma = 0); \\ X' &= \overline{X}e^{i\alpha}, \quad Y' = \overline{Y}e^{i\beta}, \quad Z' = \overline{Z}e^{i\gamma} \quad (\alpha + \beta + \gamma = 0); \end{aligned}$$

it has 2 parameters. Therefore the group G_1 has 4 parameters and the group G has 8 parameters. The group G_1 is characterized by the invariance of the variable u and therefore of the forms

$$X\overline{X} + Y\overline{Y}, \quad Z\overline{Z} + v^2, \quad (X\overline{X} - Y\overline{Y})v + XYZ + \overline{X}\ \overline{Y}\ \overline{Z}.$$

13. The groups G and G_1 are easily determined by the construction of the 4-dimensional singular variety V of the points where $F = 1$. Consider the locus of points

$$\begin{cases} X = \sqrt{3}\,\eta\,\overline{\zeta}, \quad Y = \sqrt{3}\,\zeta\,\overline{\xi}, \quad Z = \sqrt{3}\,\xi\,\overline{\eta}, \\ v = \frac{\sqrt{3}}{2}\,(\xi\,\overline{\xi} - \eta\,\overline{\eta}), \quad u = \zeta\,\overline{\zeta} - \frac{1}{2}(\xi\,\overline{\xi} + \eta\,\overline{\eta}), \end{cases} \tag{23}$$

where ξ, η, ζ are three complex parameters linked by the relation

$$\xi\,\overline{\xi} + \eta\,\overline{\eta} + \zeta\,\overline{\zeta} = 1. \tag{24}$$

It is a representative variety of the hermitian elliptical space of 4 real dimensions (two complex dimensions) with respect to the homogeneous coordinates ξ, η, ζ; indeed the second members of the formulas (23) do not change when we multiply ξ, η, ζ by the same complex factor of modulus 1. The second members are moreover the *harmonic* forms of Hermite of the second degree of the hermitan space[7].

[7] See E. Cartan, Leçons sur la géométrie projective complexe (Paris, Gauthiers-Villars, 1931), Chapter V, p. 281–322.

An easy calculation shows in the first place that $u^2+v^2+X\overline{X}+Y\overline{Y}+Z\overline{Z} = 1$, and in the second place that the form F becomes equal to 1 when we replace u, v, X, Y, Z with the values (23). The variety V defined by (23) being 4-dimensional coincides with the singular variety of the spherical space of 7 dimensions. The group G therefore contains the group induced in the 7-dimensional spherical space by the group of elliptical hermitian space. Since they are both of order 8, they are identical. *The group G is therefore isomorphic to the group of unitary unimodular substitutions with 3 variables ξ, η, ζ completed by the substitutions obtained by combining them with $\xi' = \overline{\xi}$, $\eta' = \overline{\eta}$, $\zeta' = \overline{\zeta}$.*

By setting

$$\begin{aligned} \xi' &= a\,\xi + b\,\eta + c\,\zeta, \\ \eta' &= a'\,\xi + b'\,\eta + c'\,\zeta, \\ \zeta' &= a''\,\xi + b''\,\eta + c''\zeta, \end{aligned} \tag{25}$$

where the matrix of coefficients is unitary unimodular, we would easily obtain the equations of G.

The group G_1 is the subgroup of G which leaves the variable u invariant; it comes from the subgroup of elliptic hermitian geometry which leaves $\zeta\overline{\zeta}$ invariant, that is to say the subgroup

$$\begin{cases} \xi' = e^{i\theta}(a\,\xi - b\,\eta), \\ \eta' = e^{i\theta}(\overline{b}\,\xi + \overline{a}\,\eta), \quad \text{with } a\,\overline{a} + b\,\overline{b} = 1; \\ \zeta' = e^{-2i\theta}\zeta; \end{cases} \tag{26}$$

its equations are

$$\begin{cases} X' = e^{3i\theta}(\overline{a}\;X + \overline{b}\;\overline{Y}), \\ Y' = e^{-3i\theta}(-\overline{b}\;\overline{X} + \overline{a}\;Y), \\ Z' = 2ab\;v + a^2 Z - b^2\overline{Z}, \\ v' = (a\,\overline{a} - b\,\overline{b})v - a\,\overline{b}\;Z - b\,\overline{a}\;\overline{Z}, \end{cases} \tag{27}$$

to which must be added the transformations obtained by combining the previous ones with the substitution

$$X' = \overline{X}, \quad Y' = \overline{Y}, \quad Z' = \overline{Z}, \quad v' = v. \tag{28}$$

By setting[8]

$$X = \xi_0, \quad \overline{Y} = \xi_1; \quad Z = z_1 + iz_2, \quad v = z_3,$$

equations (27) show that the group G_1 is the direct product of the one-parameter group

$$\xi_0' = e^{3i\theta}\xi_0, \quad \xi_1' = e^{3i\theta}\xi_1, \quad z_1' = z_1, \quad z_2' = z_2 \quad z_3' = z_3 \tag{29}$$

and the group

$$\begin{cases} \xi_0' = \overline{a}\ \xi_0 + \overline{b}\ \xi_1, \\ \xi_1' = -b\ \xi_0 + a\ \xi_1, \\ z_1' + iz_2' = a^2(z_1 + iz_2) - b^2(z_1 - iz_2) + 2ab\ z_3, \\ z_3' = -a\overline{b}\ (z_1 + iz_2) - b\overline{a}\ (z_1 - iz_2) + (a\overline{a} - b\overline{b})\ z_3, \end{cases} \tag{30}$$

combined with the transformation

$$\xi_0' = \overline{\xi}_0, \quad \xi_1' = \overline{\xi}_1, \quad z_1' = z_1, \quad z_2' = -z_2, \quad z_3' = z_3. \tag{31}$$

14. In the euclidean space E_7 tangent at A $(u = 1, v = X = Y = Z = 0)$ to the hypersphere of radius 1 in the euclidean space of 8 dimensions, the coordinates z_i are those of a vector of a subspace E_3 of three dimensions, and the real and imaginary parts of ξ_0 and ξ_1 are the components of a vector in a subspace E_4 of four dimensions; E_3 is the subspace normal to V at A, E_4 is the subspace tangent to V at A. We can interpret ξ_0 and ξ_1 as the complex components of a spinor in the space E_3; *the group G is then the direct product of two groups: the first leaves fixed all the vectors of the space E_3 and multiplies the spinors by a common factor of modulus 1; the second is isomorphic to the group of rotations of the space of three dimensions, acting in E_3 on the vectors, and in E_4 on the spinors of that space. One thus realizes around the point A a euclidean space of 3 dimensions and the space of 4 real dimensions of its spinors*[9].

---

[8]There is no need to confuse the quantities ξ_0, ξ_1 introduced here with the first homogeneous coordinate ξ of a point in the hermitian elliptical space.

[9]See E. Cartan, *Leçons sur la théorie des spineurs I* (Paris, Hermann, 1938). The cubic form $(X\overline{X} - Y\overline{Y})v + XYZ + \overline{X}\ \overline{Y}\ \overline{Z}$ invariant by G_1 is written with our new notation

$$(\xi_0\overline{\xi}_0 - \xi_1\overline{\xi}_1)z_3 + \xi_0\overline{\xi}_1(z_1 + iz_2) + \xi_1\overline{\xi}_0(z_1 - iz_2);$$

it represents the scalar product of the vector (z_1, z_2, z_3) by the vector $(\xi_0\overline{\xi}_1 + \xi_1\overline{\xi}_0, i\ \xi_0\overline{\xi}_1 - i\ \xi_1\overline{\xi}_0, \xi_0\overline{\xi}_0 - \xi_1\overline{\xi}_1)$ defined by the spinor (ξ_0, ξ_1) and its conjugate.

The symmetry with respect to the origin of E_3 is expressed on the spinors by the substitution for a vector (ξ_0, ξ_1) of its *conjugate* $(\overline{\xi}_1, -\overline{\xi}_0)$, as seen by combining the operation (31) with the rotation (30) with parameters $a = 0$, $b = 1$; we can deduce the effect of any inversion of E_3.

15. Note that we could go back from the variety V defined by the formulas (23) to the other isoparametric hypersurfaces of the space by the process employed in the spherical space of 4 dimensions. The locus of points located at a distance t from a point ranging over V is

$$\frac{\sqrt{3}}{2}(\eta\,\overline{\zeta}\,\overline{X} + \zeta\,\overline{\xi}\,\overline{Y} + \xi\,\overline{\eta}\,\overline{Z} + \zeta\,\overline{\eta}\,X + \xi\,\overline{\zeta}\,Y + \eta\,\overline{\xi}\,Z) + \frac{\sqrt{3}}{2}(\xi\,\overline{\xi} - \eta\,\overline{\eta})v + \left(\zeta\,\overline{\zeta} - \frac{\xi\,\overline{\xi} + \eta\,\overline{\eta}}{2}\right)u - \cos t\,(\xi\,\overline{\xi} + \eta\,\overline{\eta} + \zeta\,\overline{\zeta}) = 0.$$

The hypersurface parallel to V and at the distance t from V is obtained by canceling the discriminant of the first member considered as a form of Hermite in ξ, η, ζ. We easily find the equation

$$F = \cos 3t.$$

IV. The case of the spherical space of 13 dimensions.

16. In the case that concerns us now we have $\nu = 4$, and we introduce three quaternion variables X, Y, Z, into the form F. The form $\mathfrak{F}$ is the scalar part of the product XYZ, and the group G_2 is the group of linear substitutions that orthogonally transform the components of each of the quaternions X, Y, Z, while leaving the scalar part of their product invariant. We see immediately that the transformations

$$X' = B\;X\;\overline{C}, \quad Y' = C\;Y\;\overline{A}, \quad Z' = A\;Z\;\overline{B}, \tag{32}$$

where A, B, C are any three constant unit quaternions, $\overline{A}$, $\overline{B}$, $\overline{C}$, their conjugates (or inverses), are part of G_2 and constitute a 9-parameter subgroup of it. We have in fact

$$X'\;Y'\;Z' = B\;X\;Y\;Z\;\overline{B}, \quad \overline{Z'}\;\overline{Y'}\;\overline{X'} = B\;\overline{Z}\;\overline{Y}\;\overline{X}\;\overline{B},$$

where

$$X'\;Y'\;Z' + \overline{Z'}\;\overline{Y'}\;\overline{X'} = B(X\;Y\;Z + \overline{Z}\;\overline{Y}\;\overline{X})\;\overline{B} = X\;Y\;Z + \overline{Z}\;\overline{Y}\;\overline{X}.$$

By looking directly for the infinitesimal transformations of G_2, we notice that there are only 9 independent ones. As a result, formulas (32) give the whole group (one could easily show that G_2 is continuous and cannot be decomposed into several distinct continuous families of transformations).

The group G_2 is the direct product of three simple groups with 3 parameters, defined respectively by the parametric quaternions A, B, C.

17. The group G_1 has order $r_1 = r_2 + \nu = 9 + 4 = 13$, and the group G has order 21. The first is characterized by the invariance of u and hence the forms

$$X\ \overline{X} + Y\ \overline{Y}, \quad Z\ \overline{Z} + v^2, \quad (X\ \overline{X} - Y\ \overline{Y})\ v + X\ Y\ Z + \overline{Z}\ \overline{Y}\ \overline{X}.$$

We are going to determine it by directly determining the group G and for that we are going, as in the preceding case, to start from a direct representation of the singular variety V of 8 dimensions.

For this, we only need to start from equations (23), where ξ, η, ζ will be regarded as three quaternions linked by equation (24). Since the right side of (23) does not change when one multiplies ξ, η, ζ on the right by the same unit quaternion ϱ, the variety defined by (23) has 8 dimensions. It is easy to verify that $u^2 + v^2 + X\ \overline{X} + Y\ \overline{Y} + Z\ \overline{Z} = 1$ and that all the points of the variety (23) satisfy $F = 1$: it is therefore the singular variety V.

Note that V provides a representation of the points of the quaternionic projective space of 2 quaternionic dimensions. It is invariant by the group induced in 13-dimensional spherical space by the group of linear quaternion substitutions[10]

$$\left\{ \begin{array}{l} \xi' = A\ \xi + B\ \eta + C\ \zeta, \\ \eta' = A'\ \xi + B'\ \eta + C'\ \zeta, \\ \zeta' = A''\ \xi + B''\ \eta + C''\ \zeta, \end{array} \right. \tag{33}$$

which leave the form $\xi\ \overline{\xi} + \eta\ \overline{\eta} + \zeta\ \overline{\zeta}$ invariant. The quaternion coefficients

[10]This group cannot be extended by adding a second group of linear substitutions, as in the case of the spherical space of 7 dimensions. It is a simple closed group of the type C); it is the group of homographies, in the complex projective space of 3 dimensions, which leave invariant an anti-involution of the second kind and an elliptical anti-polarity exchangeable with this anti-involution; the points of quaternion space represent the lines of the anti-involution [See E. Cartan [7]) and S. Wachs, *Essai sur la géométrie projective quaternionienne*, Mém. Acad. de Belgique, 1936].

of the substitutions of this group are defined by the relations

$$\begin{cases} A\,\overline{A} + A'\,\overline{A'} + A''\,\overline{A''} = 1, \\ B\,\overline{B} + B'\,\overline{B'} + B''\,\overline{B''} = 1, \\ C\,\overline{C} + C'\,\overline{C'} + C''\,\overline{C''} = 1, \\ \overline{B}\,C + \overline{B'}\,C' + \overline{B''}\,C'' = 0, \\ \overline{C}\,A + \overline{C'}\,A' + \overline{C''}\,A'' = 0, \\ \overline{A}\,B + \overline{A'}\,B' + \overline{A''}\,B'' = 0. \end{cases} \tag{34}$$

By noticing that the inverse substitution of (33) is

$$\begin{cases} \xi = \overline{A}\,\xi' + \overline{A'}\,\eta' + \overline{A''}\,\zeta', \\ \eta = \overline{B}\,\xi' + \overline{B'}\,\eta' + \overline{B''}\,\zeta', \\ \zeta = \overline{C}\,\xi' + \overline{C'}\,\eta' + \overline{C''}\,\zeta', \end{cases}$$

we can deduce that the relations (34) are equivalent to the relations

$$\begin{cases} A\,\overline{A} + B\,\overline{B} + C\,\overline{C} = 1, \\ A'\,\overline{A'} + B'\,\overline{B'} + C'\,\overline{C'} = 1, \\ A''\,\overline{A''} + B''\,\overline{B''} + C''\,\overline{C''} = 1, \\ A'\,\overline{A''} + B'\,\overline{B''} + C'\,\overline{C''} = 0, \\ A''\,\overline{A} + B''\,\overline{B} + C''\,\overline{C} = 0, \\ A\,\overline{A'} + B\,\overline{B'} + C\,\overline{C'} = 0. \end{cases} \tag{35}$$

18. The transformations (33) constitute a group with 21 parameters. If we look for the infinitesimal transformations of the group (33),

$$\begin{cases} \delta\xi = a\,\xi + b\,\eta + c\,\zeta, \\ \delta\eta = a'\,\xi + b'\,\eta + c'\,\zeta, \\ \delta\zeta = a''\,\xi + b''\,\eta + c''\,\zeta, \end{cases}$$

we find that in the table of coefficients, the elements of the main diagonal have their scalar part zero, and that the elements symmetrical with respect to the main diagonal are each the negative of the conjugate of the other; this provides a number of arbitrary parameters equal to $3 \times 3 + 4 \times 3 = 21$.

We can easily deduce from (33) the equations of the group G of the spherical space of 13 dimensions, namely

$$
\left\{
\begin{array}{l}
X' = \sqrt{3}C'\ \overline{C''}u + (A'\ \overline{A''} - B'\ \overline{B''})v + B'X\overline{C''} + C'Y\overline{A''} \\
\qquad\quad +A'Z\overline{B''} + C'\ \overline{X}\ \overline{B''} + A'\ \overline{Y}\ \overline{C''} + B'\ \overline{Z}\ \overline{A''}, \\
Y' = \sqrt{3}C''\ \overline{C}u + (A''\ \overline{A} - B''\ \overline{B})v + B''X\overline{C} + C''Y\overline{A} \\
\qquad\quad +A''Z\overline{B} + C''\overline{X}\ \overline{B} + A''\overline{Y}\ \overline{C} + B''\overline{Z}\ \overline{A}, \\
Z' = \sqrt{3}C\ \overline{C'}u + (A\ \overline{A'} - B\ \overline{B'})v + BX\overline{C'} + CY\overline{A'} \\
\qquad\quad +AZ\overline{B'} + C'\ \overline{X}\ \overline{B'} + A\ \overline{Y}\ \overline{C'} + B\ \overline{Z}\ \overline{A'}, \\
v' = \frac{\sqrt{3}}{2}(C\ \overline{C} - C'\overline{C'})u + \frac{1}{2}(A\overline{A} - A'\overline{A'} - B\overline{B} + B'\overline{B'})v \\
\qquad\quad +\frac{1}{2}(BX\overline{C} - B'X\overline{C'} + C\overline{X}\ \overline{B} - C'\ \overline{X}\ \overline{B'}) \\
\qquad\quad +\frac{1}{2}(CY\overline{A} - C'Y\overline{A'} + A\overline{Y}\ \overline{C} - A'\ \overline{Y}\ \overline{C'}) \\
\qquad\quad +\frac{1}{2}(AZ\overline{B} - A'Z\overline{B'} + B\overline{Z}\ \overline{A} - B'\ \overline{Z}\ \overline{A'}), \\
u' = \left(C''\overline{C''} - \frac{A''\ \overline{A''} + B''\ \overline{B''}}{2}\right)u + \frac{\sqrt{3}}{2}(A''\ \overline{A''} - B''\ \overline{B''})v \\
\qquad\quad +\frac{\sqrt{3}}{2}(B''X\overline{C''} + C''Y\overline{A''} + A''Z\overline{B''} + C''\ \overline{X}\ \overline{B''} \\
\qquad\quad +A''\ \overline{Y}\ \overline{C''} + B''\overline{Z}\ \overline{A''}).
\end{array}
\right.
\tag{36}
$$

We find the group G_2 starting from the substitutions (33) which transform each of the quaternions into one of its multiples so that only the coefficients A, B', C'' remain.

19. The group G_1 is the one which leaves fixed the variable u, that is, comes from the subgroup of (33) which leaves fixed $\zeta\overline{\zeta}$; it is thus characterized by the relations

$$A'' = B'' = 0, \quad \text{whence} \quad C = C' = 0,$$

with

$$
\left\{
\begin{array}{ll}
A\overline{A} + A'\overline{A'} = 1, & \qquad A\overline{A} + B\overline{B} = 1, \\
B\overline{B} + B'\overline{B'} = 1, & \text{or} \quad A'\overline{A'} + B'\overline{B'} = 1, \\
\overline{A}B + \overline{A'}B' = 0; & \qquad A\overline{A'} + B\overline{B'} = 0.
\end{array}
\right.
$$

Its equations can be put in the form

$$
\left\{
\begin{array}{l}
X' = (B'X + A'\ \overline{Y})\ \overline{C''}, \\
\overline{Y'} = (BX + A\ \overline{Y})\ \overline{C''},
\end{array}
\right.
\tag{37}
$$

$$
\left\{
\begin{array}{l}
Z' = AZ\overline{B'} + B\overline{Z}\ \overline{A'} + (A\overline{A'} - B\overline{B'})\ v, \\
v' = \frac{1}{2}(AZ\overline{B} - A'Z\overline{B'} + B\overline{Z}\ \overline{A} - B'\overline{Z}\ \overline{A'}) + (A\overline{A} - A'\overline{A'})\ v.
\end{array}
\right.
\tag{38}
$$

The group G_1 is therefore the direct product of the group

$$X' = X\overline{C''}, \quad \overline{Y'} = \overline{Y}\ \overline{C''}, \quad Z' = Z, \quad v' = v,$$

and the groupe obtained by making $C'' = 1$ in the formulas (37) and (38); this last group, of order 10, is isomorphic to the group of rotations of the euclidean space of 5 dimensions whose coordinates are the components of Z and v. The equations (38) are precisely the equations of the group of rotations of this space E_5; as for the equations (37), they indicate how these rotations transform *the spinors* of this space, each spinor with 4 complex components being represented by two quaternions X and $\overline{Y}$. The cubic form[11]

$$(X\overline{X} - Y\overline{Y})\ v + XYZ + \overline{Z}\ \overline{Y}\ \overline{X}$$

is none other than the scalar product of the vector (Z, v) of E_5 by the vector defined by the spinor $(X, \overline{Y})$ and its conjugate $(\overline{X}, Y)$. The 8-dimensional space of spinors is here, in the 13-dimensional euclidean space tangent at A $(u = 1, v = X = Y = Z = 0)$ to the hypersphere of radius 1 in the 14-dimensional euclidean space, the subspace tangent to the singular variety V, while the space E_5 of vectors is the subspace normal at A to that variety. *We thus realize at the same time both the space E_5 of vectors and the space E_8 of spinors of E_5; the group G_1 of stability at the point A is decomposed in the group of rotations of E_5, with its effect on the vectors of E_5 and on the spinors on E_8, and in the group which multiplies on the right the quaternion components of a spinor by a common unit quaternion factor without altering the vectors.*

V. The case of the spherical space of 25 dimensions.

20. In this last case we have $\nu = 8$ and the quantities X, Y, Z that we introduce in the cubic form F are *octaves*[12]. Here the formulas (23) no

[11]In the book already cited [9]) of E. Cartan, *Leçons sur la théorie des spineurs,* II, the fundamental form of the space of 5 dimensions is $(x_0)^2 + x^1x^{1'} + x^2x^{2'}$ and the complex components of the spinor are designated by $\xi_0, \xi_1, \xi_2, \xi_{12}$. We pass from these notations to those of the text by setting

$$v = x^0, \quad Z = x^1 + jx^2, \quad X = \xi_0 - j\xi_{12}, \quad \overline{Y} = \xi_1 + j\xi_2$$

(the unit quaternions are $i, j, k = ij$). The cubic form of the text is none other than the cubic form $\overline{\xi}X\xi$ of the book, where X is the matrix of degree 4 associated to the vector with components $x^0, x^1, x^2, x^{1'}, x^{2'}$.

[12]See a recent memoir on octaves by E. A. Weiss: *Oktaven, Engelscher Komplex, Trialitätsprinzip* [Math. Zeitschr. **44** (1938), p. 580–611].

longer allow us to represent the 16-dimensional singular variety of the space by means of three octaves ξ, η, ζ, because the multiplication of the octaves is not associative; moreover, the second members of these formulas are altered if we multiply ξ, η, ζ on the right by the same unit octave. It is necessary to study the groups G_2, G_1 and G separately; we will leave aside the question of whether the singular variety is susceptible to a rational parametric representation.

21. The group G_2 is the group which orthogonally transforms the components of the octaves X, Y, Z while leaving invariant the scalar part of the product XYZ. Note that this scalar part is the same whether this product is interpreted as $(XY)Z$ or $X(YZ)$; nor does it change by a circular permutation of the octaves X, Y, Z. By setting,

$$X = x_0 + \sum_{i=1}^{i=7} x_i e_i, \quad Y = y_0 + \sum_{i=1}^{i=7} y_i e_i, \quad Z = z_0 + \sum_{i=1}^{i=7} z_i e_i,$$

we find[13]

$$\begin{aligned} \mathfrak{F} = x_0 y_0 z_0 \quad & - \quad x_0 \sum y_i z_i - y_0 \sum z_i x_i - z_0 \sum x_i y_i \\ & - \quad \sum x_i (y_{i+1} z_{i+3} - y_{i+3} z_{i+1} + y_{i+2} z_{i+6} - y_{i+6} z_{i+2} \\ & + \quad y_{i+4} z_{i+5} - y_{i+5} z_{i+4}), \end{aligned} \tag{39}$$

with the convention to reduce *modulo* 7 the indices greater than 7.

We can see first that if an operation of the group is reduced to the identity operation with respect to the octave Z, then it itself is reduced to either the identity operation or to the operation $X' = -X$, $Y' = -Y$, $Z' = Z$. Indeed to say that the scalar part of $(XY)Z$ is invariant whatever the octave Z, is to say that the product XY is invariant, because the scalar part of a product UV is $u_0 v_0 - u_1 v_1 - \cdots - u_7 v_7$ and its invariance for each system of values of the v_i leads to the invariance of U. Let a be the transformed octave of $X = 1$; if Y' is the transformed octave of Y[14], we will have

$$a\, Y' = Y, \quad \text{whence} \quad Y' = a^{-1}\, Y;$$

[13]The law of multiplication of the units e_i is $e_i^2 = -1$, $e_i e_{i+1} = -e_{i+1} e_i = e_{i+3}$, $e_i e_{i+2} = -e_{i+2} e_i = e_{i+6}$, $e_i e_{i+4} = -e_{i+4} e_i = e_{i+5}$, the indices greater than 7 being reduced modulo 7.

[14]The transformation law of the Y is not necessarily the same as the transformation law of the X.

moreover the modulus of a must be equal to 1, since the components x_0, x_i must undergo an orthogonal substitution, so we can write

$$Y' = \overline{a}\, Y \quad (a\overline{a} = 1) : \tag{40}$$

we can deduce

$$X'(\overline{a}\, Y) = XY,$$

and, by making $Y = 1$,

$$X' = Xa. \tag{41}$$

The octave a must therefore satisfy the identity

$$(Xa)(\overline{a}Y) = (a\overline{a})(XY)$$

whatever the octaves X, Y; in particular, we must have

$$(e_1 a)(\overline{a} e_2) = a\overline{a} e_4;$$

the coefficient of e_4 in the first member is

$$a_0^2 - \sum (e_1 e_i)(e_i e_2) a_i^2 = a_0^2 + a_1^2 + a_2^2 + a_4^2 - a_3^2 - a_5^2 - a_6^2 - a_7^2;$$

by equating it to the coefficient $a\overline{a}$ of e_4 in the second member, we obtain

$$a_3^2 + a_5^2 + a_6^2 + a_7^2 = 0, \quad \text{whence} \quad a_3 = a_5 = a_6 = a_7 = 0;$$

we would likewise show $a_1 = a_2 = a_4 = 0$. Consequently, $a = \pm 1$, which had to be shown.

From the preceding, we get that to any orthogonal substitution on the components of Z there correspond at most two orthogonal substitutions on the components of X and Y, that is, two transformations of G_2. Therefore, the group G_2 has at most $\frac{8 \cdot 7}{1 \cdot 2} = 28$ parameters.

22. The calculation actually shows that G_2 has 28 parameters and provides the infinitesimal transformations, which we will only indicate. By setting

$$X_{\alpha\beta} = x_\alpha \frac{\partial f}{\partial x_\beta} - x_\beta \frac{\partial f}{\partial x_\alpha},\ Y_{\alpha\beta} = y_\alpha \frac{\partial f}{\partial y_\beta} - y_\beta \frac{\partial f}{\partial y_\alpha},\ Z_{\alpha\beta} = z_\alpha \frac{\partial f}{\partial z_\beta} - z_\beta \frac{\partial f}{\partial z_\alpha},$$

these infinitesimal transformations are

$$\begin{cases} U_{0i} \equiv Z_{0i} - \frac{1}{2}(X_{i+1,i+3} + X_{i+2,i+6} + X_{i+4,i+5}) \\ \qquad\quad + \frac{1}{2}(Y_{i+1,i+3} + Y_{i+2,i+6} + Y_{i+4,i+5}), \\ U_{i+1,i+3} \equiv Z_{i+1,i+3} + \frac{1}{2}(X_{0i} + X_{i+1,i+3} - X_{i+2,i+6} - X_{i+4,i+5}) \\ \qquad\quad + \frac{1}{2}(-Y_{0i} + Y_{i+1,i+3} - Y_{i+2,i+6} - Y_{i+4,i+5}), \\ U_{i+2,i+6} \equiv Z_{i+2,i+6} + \frac{1}{2}(X_{0i} - X_{i+1,i+3} + X_{i+2,i+6} - X_{i+4,i+5}) \\ \qquad\quad + \frac{1}{2}(-Y_{0i} - Y_{i+1,i+3} + Y_{i+2,i+6} - Y_{i+4,i+5}), \\ U_{i+4,i+5} \equiv Z_{i+4,i+5} + \frac{1}{2}(X_{0i} - X_{i+1,i+3} - X_{i+2,i+6} + X_{i+4,i+5}) \\ \qquad\quad + \frac{1}{2}(-Y_{0i} - Y_{i+1,i+3} - Y_{i+2,i+6} + Y_{i+4,i+5}); \end{cases} \tag{42}$$

the index i takes all the values $1, \ldots, 7$, each index greater than 7 being reduced *modulo* 7.

In 26-dimensional euclidean space, the euclidean space of 24 dimensions tangent at the point P $(u = \cos t, v = \sin t, X = Y = Z = 0)$ to the hypersurface with parameter t situated in the sphere of radius 1 decomposes into three 8-dimensional subspaces, that of the vectors X, of the vectors Y, and of the vectors Z. If we turn our attention to the latter, the vectors X of the first can be regarded as the *real semi-spinors of the first kind*, and the vectors Y of the second as the *real semi-spinors of the second kind* of the space of the Z[15]. The group G_2 indicates how the group of rotations of the 8-dimensional space transforms the real vectors Z, the real semi-spinors of the first kind X, and the semi-spinors of the second kind Y. *The principle of triality*[16] *of the elliptical space of 7 dimensions is thus demonstrated in a concrete way.*

23. The group G_1 has $28 + 8 = 36$ parameters, and the group G has $36 + 16 = 52$ parameters. First we will take care of the group G_1. It

[15] See the work [9]) of E. Cartan already cited. In the real euclidean space E_8, the semi-spinors of the first kind have the components $\xi_0, \xi_{ij}, \xi_{1234}$, those of the second kind have the components $\xi_i, \xi_{ijk} (i, j, k = 1, 2, 3, 4)$. For each kind there is a *domain of reality*; the real semi-spinors of the first kind are defined by $\xi_{1234} = \bar{\xi}_0$, $\xi_{14} = -\bar{\xi}_{23}$, $\xi_{24} = -\bar{\xi}_{31}$, $\xi_{34} = -\bar{\xi}_{12}$, the scalar square of a semi-spinor being $\xi_0\bar{\xi}_0 + \xi_{23}\bar{\xi}_{23} + \xi_{31}\bar{\xi}_{31} + \xi_{12}\bar{\xi}_{12}$; the real semi-spinors of the second kind are defined by $\xi_{234} = -\bar{\xi}_1$, $\xi_{314} = -\bar{\xi}_2$, $\xi_{124} = -\bar{\xi}_3$, $\xi_{123} = -\bar{\xi}_4$, the scalar square being $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3 + \xi_4\bar{\xi}_4$. The trilinear form $\mathfrak{F}$ is indicated in the book in question, II, p. 51. The group G_2 can be completed by five other families of linear substitutions, each family performing a certain permutation on the three families of objects: vectors, semi-spinors of the first kind, semi-spinors of the second kind. As far as the problem of the text is concerned, none of these five families has to intervene.

[16] Apart from the work cited in the preceding note, see the previously mentioned Memoir [12]) of E. A. Weiss, p. 592–596.

orthogonally transforms the vectors of the space E_{16} tangent to the singular variety V at the point A $(u = 1, v = X = Y = Z = 0)$ and the vectors of the space E_9 normal to that variety. The space E_{16} is defined by $Z = v = 0$ and the space E_9 by $X = Y = 0$. The group G_1 orthogonally transforms the vectors of E_9 according to the group of rotations of that space. As for the vectors of E_{16}, they are transformed like the real spinors of E_9. The spinors of the euclidean space of 9 dimensions have 16 complex components, but within the space of such spinors there exists a domain of reality *(domaine de réalité)* within which the spinors have 16 real components[17]. The fact that G_1 transforms the vectors of E_{16} as the real spinors of the 9-dimensional euclidean space has the a priori reason that the only linear representation of degree 16 of the group of rotations of E_9 is that of the spinors.

We therefore have around the point A both the space E_9 of real vectors and the space E_{16} of real spinors of the euclidean space of dimension 9. The group G_1 of stability at the point A in the spherical space of 25 dimensions is none other than the group of rotations of this last space, with its effect on the vectors in E_9 and on the spinors in E_{16}.

24. Finally, we consider the group G. We are going to show that G is a simple group of type F). First the group G is closed, since it is the largest orthogonal group with 26 variables leaving invariant an entire algebraic form of the variables. Now every closed group is either simple or semi-simple (direct product of several simple groups), or abelian, or direct product of a simple group or a semi-simple group and an abelian group[18]. The rank of a closed group is the order of any maximal abelian subgroup, the rank of an abelian group being therefore identical to its order. It is clear that here the group G is not abelian.

To calculate the rank of G, first note that the group G_2, isomorphic to the orthogonal group with 8 variables, has rank 4; one can moreover take in G_2 the abelian group γ generated by the infinitesimal transformations $U_{01}, U_{24}, U_{37}, U_{56}$; this group is abelian because these transformations only involve the 12 infinitesimal transformations $X_{01}, X_{24}, X_{37}, X_{56}, Y_{01}, \ldots, Z_{56}$,

[17] The complex components of a spinor of E_9 can be denoted $\xi_0, \xi_i, \xi_{ij}, \xi_{ijk}, \xi_{1234}$ $(i, j, k = 1, 2, 3, 4)$. The real spinors are defined by the conditions $\xi_{1234} = \bar{\xi}_0$, $\xi_{123} = \bar{\xi}_4$, $\xi_{234} = -\bar{\xi}_1$, $\xi_{314} = -\bar{\xi}_2$, $\xi_{124} = -\bar{\xi}_3$, $\xi_{14} = -\bar{\xi}_{23}$, $\xi_{24} = -\bar{\xi}_{31}$, $\xi_{34} = -\bar{\xi}_{12}$; the scalar square of a real spinor is $\xi_0\bar{\xi}_0 + \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3 + \xi_4\bar{\xi}_4 + \xi_{23}\bar{\xi}_{23} + \xi_{31}\bar{\xi}_{31} + \xi_{12}\bar{\xi}_{12}$.

[18] For all these properties, see E. Cartan, *La théorie des groupes finis et continus et l'Analysis Situs* (Mem. Sc. Math. XLII, 1930).

obviously exchangeable among themselves. That being so, any infinitesimal transformation of G exchangeable with γ will transform any one of the variables u, v, invariant under γ, into a combination of the 26 variables which will also be invariant by γ. Now γ obviously admits no other invariant linear combination than u and v. Consequently, the infinitesimal transformation sought will transform the variables u and v among themselves; consequently, being orthogonal, it will transform the other 24 variables among themselves. The collection $u^3 - 3uv^2$ of terms of F which are of third degree in u, v will thus also be invariant: this is only possible if each of the variables u, v is invariant. Hence any transformation of G exchangeable with γ is part of G_2; γ is thus a maximal abelian subgroup of G, which is therefore of rank 4.

Now the simple groups of rank ≤ 4 have for possible orders

rank 4: orders 24, 28, 36, 52; rank 2: orders 8, 10;

rank 3: orders 15, 21; rank 1[19]: orders 1, 3.

This table immediately shows that a closed group of rank 4 can only have order 52 if it is simple and therefore of type F).

The group G of displacements of the family under consideration of isoparametric hypersurfaces of the spherical space of 25 dimensions is thus a realization of the simple group of rank 4 with 52 parameters of the type F); until now this group had not appeared in any problem of Geometry or Analysis[20].

We will leave aside the effective determination of its 52 infinitesimal transformations to the variables u, v, x_i, y_i, z_i $(i = 0, 1, 2, \ldots, 7)$.

VI. A theorem on the singular varieties.

25. The singular varieties of 2 and 4 dimensions in the spherical spaces of 4 and 7 dimensions are, considered as riemannian varieties, *symmetric*[21]

[19]Here we regard the closed abelian group with one parameter as a simple group.

[20]The representative indicated in the Thesis of E. Cartan is indeed an orthogonal linear group with 26 variables, but of rather complicated definition, leaving invariant a variety of 16 dimensions.

[21]On the notion of a riemannian symmetric space, see a lecture by E. Cartan at the International Congress of Mathematicians in Zürich in 1932.

spaces, since they are applicable on the ordinary sphere and on the elliptical hermitien space. A simple argument shows that this property of being symmetric also holds for the singular varieties of 8 and 16 dimensions in the spherical spaces of 13 and 25 dimensions. Indeed let A be the point $(u = 1, v = z_i = 0)$ of one of these varieties. We consider in the euclidean space of $n+1 = 3\nu+2$ dimensions which contains the hypersphere of radius 1 of n dimensions, the symmetry

$$u' = u, \quad v' = v, \quad z_i' = z_i, \quad x_i' = -x_i, \quad y_i' = -y_i$$

with respect to the plane variety $x_i = y_i = 0$ normal to the variety V. This symmetry obviously leaves this variety V invariant, as is evident from the expression of the form F. This transformation is involutive on V; it obviously preserves the metric of V; finally it admits the point A as an isolated invariant point, because the points of V invariant by this transformation are defined by the numerical values of u, v, z_i which satisfy the relations

$$u^2 + v^2 + \sum z_i^2 = 1, \quad u^3 - 3u\,(v^2 + \sum z_i^2) = 1,$$

equivalent to

$$4u^3 - 3u = 1, \quad v^2 + \sum z_i^2 = 1 - u^2.$$

The equation in u has for roots $u = 1$ and $u = -\frac{1}{2}$; to the first root $u = 1$ correspond the values $v = z_i = 0$ (isolated invariant point); to the second $u = -\frac{1}{2}$ correspond an infinite number of values v and z_i satisfying the unique relation $v^2 + \sum z_i^2 = \frac{3}{4}$.

The previous properties of the transformation under consideration show that in the riemannian space constituted by the variety V, the symmetry with respect to A is isometric. As the riemannian variety V admits a transitive group of displacements, it follows that this variety is symmetric, whence the

Theorem. *In the family of isoparametric hypersurfaces with three distinct principal curvatures, the two singular varieties constitute two riemannian symmetric spaces whose groups of displacements have respectively* 3, 8, 21 *and* 52 *parameters*[22].

The nonsingular hypersurfaces *do not constitute* riemannian symmetric varieties.

[22] The variety of 8 dimensions is a symmetric space of the type C (II) and the variety of 16 dimensions is a symmetric space of the type F (II): on the geodesics of the latter, see E. Cartan [Annales Ecole Norm **44** (1927), No. 151, p. 466].

Second Part.

VII. The non-existence of families of isoparametric hypersurfaces with three distinct principal curvatures not having the same degrees of multiplicity.

26. We propose in this second Part to show that we have obtained all the families of isoparametric hypersurfaces with three distinct principal curvatures.

Let us take again the notations of the Memoir already cited and start from the formula

$$\omega_{ij} = \gamma_{ijk}\omega_k, \tag{[2]}$$

the summation index k varying from 1 to n, the dimension of the ambient space. We have, for $k \leq n-1$ and supposing that the principal curvatures a_i and a_j are distinct,

$$\gamma_{ijk} = \frac{\lambda_{ijk}}{a_i - a_j}; \tag{43}$$

λ_{ijk} is symmetric with respect to its three indices and can only be different from zero if the principal curvature a_k is distinct from both a_i and a_j (p. 179–180 of the Memoir); we also have $\gamma_{ijn} = 0$. We call the three principal curvatures $a = \tan t_1, b = \tan t_2, c = \tan t_3$, and in what follows, we will reserve the letters $i, i', \ldots,$ for the indices for which $a_i = a$, the letters $j, j', \ldots,$ for the indices for which $a_j = b$, the letters $k, k', \ldots,$ for the indices for which $a_k = c$.

By taking the exterior derivative of [2], we get

$$[\omega_{iu}\ \omega_{uj}] - (1 + a_i a_j)[\omega_i\ \omega_j] = \frac{\lambda_{ijk}}{a_i - a_j}[\omega_u\ \omega_{uk} + a_k\omega_k\ \omega_n] + [d\gamma_{ijk}\ \omega_k], \tag{44}$$

the summation index k having the meaning that has just been indicated. By equating in (44) the terms in $[\omega_i\ \omega_j]$, we obtain the formula (11) of the Memoir cited, which can be written

$$2\cos^2 t_1 \cos^2 t_2 \cos^2 t_3 \sum_k \lambda_{ijk}^2 = \sin(t_1 - t_3)\sin(t_2 - t_3)\cos(t_1 - t_2) = 2h^2. \tag{45}$$

Now by equating the terms in $[\omega_i\ \omega_{j'}]$ where $j' \neq j$, we obtain

$$\sum_k \lambda_{ijk}\lambda_{ij'k}\left[\frac{1}{(a_i - a_k)(a_j - a_k)} - \frac{1}{(a_i - a_j)(a_i - a_k)} + \frac{1}{(a_i - a_j)(a_j - a_k)}\right] = 0,$$

and, since the quantity in square brackets is nonzero,

$$\sum_k \lambda_{ijk}\lambda_{ij'k} = 0 \quad \text{and likewise} \quad \sum_k \lambda_{ijk}\lambda_{i'jk} = 0. \tag{46}$$

Finally, by equating the terms in $[\omega_{i'}\ \omega_{j'}]$, where $i' \neq i$, $j' \neq j$, we find

$$\sum_k (\lambda_{ijk}\lambda_{i'j'k} + \lambda_{ij'k}\lambda_{i'jk}) = 0. \tag{47}$$

We will simplify the equations (45) a little by setting

$$c_{ijk} = \frac{1}{h}\cos t_1 \cos t_2 \cos t_3\ \lambda_{ijk}; \tag{48}$$

the formulas (45), (46), (47) then become

$$\textstyle\sum_k c_{ijk}^2 = 1. \tag{45$'$}$$

$$\textstyle\sum_k c_{ijk}\ c_{ij'k} = 0, \quad \sum_k c_{ijk}\ c_{i'jk} = 0, \tag{46$'$}$$

$$\textstyle\sum_k (c_{ijk}\ c_{i'j'k} + c_{ij'k}\ c_{i'jk}) = 0. \tag{47$'$}$$

Moreover by equating the terms in $[\omega_k\ \omega_n]$ in (44), we find that *the* c_{ijk} *are independent of* t.

27. The preceding formulas will show us immediately the the degrees of multiplicity ν_1, ν_2, ν_3 of the principal curvatures a, b, c are equal to each other. Indeed suppose, without loss of generality, that

$$\nu_1 \geq \nu_3, \quad \nu_2 \geq \nu_3.$$

If we fix the indices i and j, the ν_3 quantities c_{ijk} can be regarded as the components of a vector $\vec{c}_{ij}$ in a space of ν_3 dimensions; by (45$'$) this vector is a unit vector. If we fix the index i, the ν_2 vectors $\vec{c}_{i1}, \vec{c}_{i2}, \ldots, \vec{c}_{i\nu_2}$ are, by (45$'$) and (46$'$), ν_2 rectangular unit vectors. This is only possible if $\nu_2 \leq \nu_3$; thus we necessarily have $\nu_2 = \nu_3$. We could prove likewise that $\nu_1 = \nu_3$. Therefore, the three degrees of multiplicity ν_1, ν_2, ν_3 are equal to each other.

VIII. Direct search for families of isoparametric hypersurfaces with three distinct principal curvatures[23].

28. Consider two systems of ν variables x_i, y_i and the ν bilinear forms

$$H_k(x,y) = \sum_{i,j} c_{ijk} x_i y_j;$$

the relations (45′), (46′), (47′) simply express the identity (14) of N° 4:

$$\textstyle\sum_k H_k^2 = \sum x_i^2 \cdot \sum y_i^2. \tag{14}$$

It is precisely from this identity that we have deduced the only possibilities $\nu = 1, 2, 4, 8$. The trilinear form

$$\Phi = \sum_{i,j,k} c_{ijk} x_i y_j z_k$$

is analogous to the form that we had designated by $\mathfrak{F}$. The research on the form F had shown us (N° 6) that we can always perform on each of the series of ν variables x_i, y_j, z_k an orthogonal linear substitution so that the coefficients of the form $\mathfrak{F}$ become certain well determined constants. We are going to deduce that here we can arrange for the coefficients c_{ijk} to have those same constant values by a suitable choice of the rectangular frames attached at the various points of the space.

Indeed let us start from the identity

$$\sum_{i,j} \omega_i \omega_j \omega_{ij} = \sum_{i,j,k} \frac{\lambda_{ijk}}{a_i - a_j}\ \omega_i \omega_j \omega_k = \frac{h}{\sin(t_1 - t_2)\cos t_3} \sum_{i,j,k} c_{ijk}\ \omega_i \omega_j \omega_k.$$

Let us carry out on the ν unit vectors $\vec{e}_i$ of indices $i, i', \ldots,$ of the frame an infinitesimal linear substitution

$$\delta \vec{e}_i = \sum_{i'} a_{ii'} \vec{e}_{i'} \quad (a_{ii'} = -a_{i'i})$$

and similarly on the ν unit vectors $\vec{e}_j$ the substitution

$$\delta \vec{e}_j = \sum_{i'} b_{jj'} \vec{e}_{j'} \quad (b_{jj'} = -b_{j'j});$$

[23] The paragraph VIII can be passed by the reader, the considerations of paragraph IX leading to the same results by a simpler route.

we will have

$$\begin{aligned}\delta\omega_{ij} = \delta(d\vec{e}_i\ \vec{e}_j) &= \sum_{i'}(da_{ii'}\ \vec{e}_{i'} + a_{ii'}\ d\vec{e}_{i'})\ \vec{e}_j + \sum_{j'} d\vec{e}_i \cdot b_{jj'}\ \vec{e}_{j'} \\ &= \sum_{i'} a_{ii'}\ \omega_{i'j} + \sum_{j'} b_{jj'}\ \omega_{ij'}\ ;\end{aligned}$$

on the other hand we have

$$\delta\omega_i = \sum_{i'} a_{ii'}\ \omega_{i'}, \quad \delta\omega_j = \sum_{j'} b_{jj'}\ \omega_{j'}\ ;$$

we can deduce

$$\delta\sum_{i,j}\omega_i\omega_j\omega_{ij} = 0.$$

This amounts to saying that the cubic form $\sum_{i,j,k} c_{ijk}\omega_i\omega_j\omega_k$ has a geometric meaning independent of the choice of the ν unit vectors $\vec{e}_i$, the ν unit vectors $\vec{e}_j$, the ν unit vectors $\vec{e}_k$; one can consequently choose these unit vectors so as to reduce the coefficients to have the fixed numerical values of the coefficients of the form $\mathfrak{F}$ determined in the first part. *That is what we will assume in the following.*

29. An important conclusion can be drawn from this. Let us return to the relation (44) and keep in this relation only the terms which contain at least one of the terms ω_k as a factor. By passing everything into the first member, these terms reduce to

$$\begin{aligned}&\sum_{i'}[\omega_{ii'}\ \omega_{i'j}] + \sum_{j'}[\omega_{ij'}\ \omega_{j'j}] - \sum_k \frac{\lambda_{ijk}}{a_i - a_j}\sum_{k'}[\omega_{k'}\ \omega_{k'k}] \\ &= \sum_k\left[\left(\sum_{i'}\frac{\lambda_{i'jk}}{a_i - a_j}\omega_{ii'} + \sum_{j'}\frac{\lambda_{ij'k}}{a_i - a_j}\omega_{jj'} + \sum_{k'}\frac{\lambda_{ijk'}}{a_i - a_j}\omega_{kk'}\right)\omega_k\right].\end{aligned}$$

It follows that for each system of indices i, j, k, the sum

$$\sum_{i'} c_{i'jk}\omega_{ii'} + \sum_{j'} c_{ij'k}\omega_{jj'} + \sum_{k'} c_{ijk'}\omega_{kk'}$$

cannot depend on the forms ω_i nor on the forms ω_j; for reasons of symmetry, it cannot depend on the forms ω_k either; it is therefore identically zero. We then have the important identity

$$\sum_{i'} c_{i'jk}\omega_{ii'} + \sum_{j'} c_{ij'k}\omega_{jj'} + \sum_{k'} c_{ijk'}\omega_{kk'} = 0. \tag{49}$$

Note that this identity holds whatever rectangular frames are chosen, *provided that they give the c_{ijk} the fixed numerical values indicated.* If we can attach to each point an infinity of reference frames satisfying this condition, the forms $\omega_{ii'}, \omega_{jj'}, \omega_{kk'}$ involve auxiliary parameters and by varying only these auxiliary parameters, the identities (49) express the invariance of the trilinear form $\sum c_{ijk}\omega_i\omega_j\omega_k$.

30. We are now able to demonstrate that the only families of isoparametric hypersurfaces with three distinct principal curvatures are those that were determined in the first part. Indeed consider a point A of the singular variety V defined by $t_3 = \frac{\pi}{2}$; we can suppose $t_2 = \frac{\pi}{2} - \frac{2\pi}{3},\ t_1 = \frac{\pi}{2} + \frac{2\pi}{3}$, whence $h = \frac{\sqrt{3}}{4}$. At the point A is attached a rectangular frame with origin at A; with respect to this frame, let

$$x_i, y_j, z_k, v, u$$

be the normal coordinates of Weierstrass of a point M of the ambient spherical space, the point M being

$$M = u\ A + \sum x_i\ \vec{e}_i + \sum y_j\ \vec{e}_j + \sum z_k\ \vec{e}_k + v\ \vec{e}_n.$$

Consider the polynomial

$$\begin{aligned} F \equiv u^3 - 3uv^2 \ &+\ \frac{3}{2}u\left(\sum x_i^2 + \sum y_j^2 - 2\sum z_k^2\right) \\ &+\ \frac{3\sqrt{3}}{2}v\left(\sum x_i^2 - \sum y_j^2\right) - 3\sqrt{3}\sum c_{ijk}x_iy_jz_k. \end{aligned}$$

When the point A of V is varied, *the point M remaining fixed*, the relative coordinates x_i, y_j, z_k, v, u of M vary. We are going to show that *the function F of these coordinates retains its numerical value.* Suppose that is done. We see first that any point of V gives the value 1 to the polynomial F since the relative coordinates of this point in the frame attached to it are $u = 1, v = x_i = y_j = z_k = 0$. Now let M be a point of the hypersurface located at a distance t from V, and let A be the point where the orthogonal trajectory of the isoparametric hypersurfaces which passes through M passes through V; if we relate the point M to the frame with origin A, its coordinates x_i, y_j are zero, and we have moreover

$$u = \cos t, \quad \sum_k z_k^2 + v^2 = \sin^2 t;$$

we can deduce

$$F = \cos^3 t - 3\cos t \sin^2 t = \cos 3t.$$

From this we get that the general equation of the isoparametric hypersurfaces is

$$F = \cos 3t.$$

31. Everything therefore amounts to demonstrating the invariance of the numerical value that the polynomial F takes at a fixed point M in space when we vary the frames to which this point is referred, these frames having their origin on the variety V. However, when the frame varies, the relative coordinates of a fixed point satisfy the relations which follow from

$$d\,(u\,A + x_i\,\vec{e}_i + y_j\,\vec{e}_j + z_k\,\vec{e}_k + v\,\vec{e}_n) = 0.$$

These relations are

$$\left\{\begin{array}{l} du = \sum x_i\omega_i + \sum y_j\omega_j + \sum z_k\omega_k + v\,\omega_n, \\ dv = \sum x_i\omega_{ni} + \sum y_j\omega_{nj} + \sum z_k\omega_{nk} - u\,\omega_n, \\ dx_i = \sum_{i'} x_{i'}\omega_{ii'} + \sum_j y_j\omega_{ij} + \sum_k z_k\omega_{ik} + v\,\omega_{in} - u\,\omega_i, \\ dy_j = \sum_i x_i\omega_{ji} + \sum_{j'} y_{j'}\omega_{jj'} + \sum_k z_k\omega_{jk} + v\,\omega_{jn} - u\,\omega_j, \\ dz_k = \sum_i x_i\omega_{ki} + \sum_j y_j\omega_{kj} + \sum_{k'} z_{k'}\omega_{kk'} + v\,\omega_{kn} - u\,\omega_k. \end{array}\right. \tag{50}$$

Here we are on the variety V. To see clearly how things happen, let us set in a general way, at any point in space,

$$\left\{\begin{array}{lll} \omega_i\ = 2\cos t_1\,\widetilde{\omega}_i, & \omega_j\ = 2\cos t_2\,\widetilde{\omega}_j, & \omega_k\ = 2\cos t_3\,\widetilde{\omega}_k, \\ \omega_{in} = 2\sin t_1\,\widetilde{\omega}_i, & \omega_{jn} = 2\sin t_2\,\widetilde{\omega}_j, & \omega_{kn} = 2\sin t_3\,\widetilde{\omega}_k, \end{array}\right. \tag{51}$$

whence

$$\omega_{jk} = \sum_i \frac{2\lambda_{jki}\cos t_1 \cos t_2 \cos t_3}{\sin(t_2 - t_3)}\,\widetilde{\omega}_i,$$

and, simplifying,

$$\left\{\begin{array}{l} \omega_{jk} = \sum_i c_{ijk}\,\widetilde{\omega}_i, \\ \omega_{ki} = \sum_j c_{ijk}\,\widetilde{\omega}_j, \\ \omega_{ij} = \sum_k c_{ijk}\,\widetilde{\omega}_k. \end{array}\right. \tag{52}$$

Given this, the formulas (50) become, by replacing respectively $\omega_i, \omega_j, \omega_k$ by $-\sqrt{3}\,\widetilde{\omega}_i, \sqrt{3}\,\widetilde{\omega}_j, 0$ and $\omega_{in}, \omega_{jn}, \omega_{kn}$ by $-\widetilde{\omega}_i, -\widetilde{\omega}_j, 2\,\widetilde{\omega}_k$, and finally ω_n by 0,

$$\left\{\begin{array}{l} du = -\sqrt{3}\ \sum x_i\,\widetilde{\omega}_i + \sqrt{3}\ \sum y_j\,\widetilde{\omega}_j, \\ dv = \sum x_i\,\widetilde{\omega}_i + \sum y_j\,\widetilde{\omega}_j - 2\ \sum z_k\,\widetilde{\omega}_k \\ dx_i = \sum_{i'} x_{i'}\omega_{ii'} + \sum_{j,k} c_{ijk}(y_j\,\widetilde{\omega}_k - z_k\,\widetilde{\omega}_j) - v\,\widetilde{\omega}_i + u\sqrt{3}\,\widetilde{\omega}_i, \\ dy_j = \sum_{j'} y_{j'}\omega_{jj'} + \sum_{k,i} c_{ijk}(z_k\,\widetilde{\omega}_i - x_i\,\widetilde{\omega}_k) - v\,\widetilde{\omega}_j - u\sqrt{3}\,\widetilde{\omega}_j, \\ dz_k = \sum_{k'} z_{k'}\omega_{kk'} + \sum_{i,j} c_{ijk}(x_i\,\widetilde{\omega}_j - y_j\,\widetilde{\omega}_i) + 2\ v\,\widetilde{\omega}_k. \end{array}\right. \qquad (50')$$

A fairly long, but not difficult, calculation then shows that taking relations (45′), (46′), (47′) and (49) into account, the differential of the polynomial F, when the differentials of the variables are given by the formulas (50′), is identically zero. *The theorem is thus proved.*

IX. Families of isoparametric hypersurfaces whose principal curvatures all have the same degree of multiplicity.

32. The verification calculations made in paragraph VIII can be avoided by resorting to a general theorem on the families of isoparametric hypersurfaces whose distinct principal curvatures, of any number p, all have the same degree of multiplicity. This theorem is the following.

Theorem. *If in a spherical space of n dimensions, there exists a family of isoparametric hypersurfaces with p distinct principal curvatures of the same degree of multiplicity ν, the general equation of these hypersurfaces is of the form*

$$P(x_1, x_2, \ldots, x_{n+1}) = \cos pt,$$

where P is a harmonic polynomial of degree p satisfying the condition

$$\sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 = p^2(x_1^2 + x_2^2 + \cdots + x_{n+1}^2)^{p-1}.$$

Note first that we can always assume that the sum of the p distinct principal curvatures is equal to $-p\cot pt$. Indeed[24] these curvatures are of the form $\tan(t + \frac{i\pi}{p} + \alpha)$, $(i = 0, 1, 2, \ldots, p-1)$, α being a constant, otherwise

[24]Annali di Mat. **17** (1938), p. 187.

arbitrary. These are the p roots of the equation which gives $\tan x$ knowing $\tan px = \tan(pt + p\alpha)$. Now the sum of the roots of this equation of degree p is $p \tan px$ if p is odd, and $-p \cot px$ if p is even, as follows immediately from the formula that gives $\tan px$ as a rational function of $\tan x$. It is then enough to take $\alpha = 0$ for p even, and $\alpha = \frac{\pi}{2p}$ for p odd, to arrive at the stated result.

Given this, we have seen[25] that the first two differential parameters of t, calculated in the spherical space of n dimensions, are

$$\Delta_1 t = 1, \quad \Delta_2 t = -(a_1 + a_2 + \cdots + a_{n-1}) = \nu \; p \cot pt. \tag{53}$$

The function $V = \cos pt$ then satisfies the relations

$$\left\{ \begin{array}{l} \Delta_1 V = p^2 \sin^2 pt = p^2(1 - V^2), \\ \Delta_2 V = -p^2 \cos pt - p \sin pt \; \Delta_2 t = -p^2(\nu + 1)V = -p(p + n - 1)V. \end{array} \right. \tag{54}$$

In the euclidean space of $n+1$ dimensions whose hypersphere of radius 1 realizes the spherical space under consideration, we now form the function

$$P(x_1, x_2, \ldots, x_{n+1}) = r^p \cos pt = r^p V, \tag{55}$$

in which r designates the distance of the point (x_i) to the origin, and t is the value of the function t at the point $\frac{x_i}{r}$ of the hypersphere. We can set

$$dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2 = dr^2 + r^2(\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2),$$

by designating by $\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2$ the ds^2 of the hypersphere of radius 1. We have

$$dP = p \; r^{p-1} V dr + \sum_i r^{p-1} V_i \; r \; \omega_i. \tag{56}$$

The laplacian of the function $P = r^p V$ can be calculated by the formula

$$\begin{aligned} \int_{(n)} p r^{p-1} V r^n \omega_1 \omega_2 \ldots \omega_n \quad & - \quad r^{p-1} V_1 \; dr \cdot r^{n-1} \omega_2 \omega_3 \ldots \omega_n \\ & + \quad r^{p-1} V_2 \; dr \cdot r^{n-1} \omega_1 \omega_3 \ldots \omega_n - \ldots \\ & = \quad \int_{(n+1)} \sum \frac{\partial^2 P}{\partial x_i^2} \cdot dr \; r^n \omega_1 \omega_2 \ldots \omega_n, \end{aligned}$$

[25] l. c. [1]), p. 178. We have $dt = \omega_n$ and, in the formula at the bottom of page 178, applied to $f = t$, we have $\varphi = 1, \psi = 0$.

the first member being an n-ple integral extended to the boundary of a domain of the euclidean space, and the $(n+1)$-ple integral to this domain itself.

The application of the generalized Stokes formula (or methods of exterior calculus) give

$$\sum_i \frac{\partial^2 P}{\partial x_i^2} = p\ (p-1+n)\ r^{p-2}\ V + r^{p-2}\ \Delta_2 V = 0. \tag{57}$$

On the other hand, we have

$$\sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 = p^2\ r^{2p-2} V^2 + r^{2p-2} \Delta_1 V = p^2 r^{2p-2}. \tag{58}$$

33. It is important to note that we do not know a priori the existence of a *regular* family of hypersurfaces. We simply assume the *local* existence, in a certain domain of spherical space, of a function t satisfying equations (54); this results in the existence in a certain domain of euclidean space of a function P satisfying equations (57) and (58). This function being harmonic is *analytic* and thus infinitely differentiable. We are going to prove that it is a polynomial of degree p[26].

For this we are going to apply the operation $\sum_i \frac{\partial^2}{\partial x_i^2}$ several times in a row to the two members of the relation (58). Notice

1° that this operation applied to r^{2q} gives r^{2q-2} up to a positive constant factor;

2° that this operation applied to the function

$$\sum_{i_1, i_2, \ldots, i_q} \left(\frac{\partial^q P}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_q}}\right)^2$$

gives, up to a positive factor, the analogous function where q is increased by one unit: an easy calculation, based on the remark that P is harmonic as well as all its derivatives, shows this result.

[26] If we knew a priori that the function t is defined on the whole spherical space, we could demonstrate without further calculation that V being one of the functions which generalizes the spherical functions, P is indeed a polynomial of degree p.

Given this, the p times repeated application of the operation $\sum_i \frac{\partial^2}{\partial x_i^2}$ to the two members of equation (58) gives

$$\sum_{i_1, i_2, \ldots, i_{p+1}} \left(\frac{\partial^{p+1} P}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_{p+1}}} \right)^2 = 0,$$

from which results the conclusion that P is an entire polynomial of degree p (and certainly not of lesser degree). The theorem is thus demonstrated.

This result proves that all the families of hypersurfaces considered are *regular*.

34. The theorem of N° 32 has a reciprocal. Let P be a harmonic polynomial of degree p satisfying the relation

$$\sum_i \left(\frac{\partial P}{\partial x_i} \right)^2 = p^2 (x_1^2 + x_2^2 + \cdots + x_{n+1}^2)^{p-1}.$$

An easy calculation gives, denoting by $\Delta_1 V$ and $\Delta_2 V$ the differential parameters, in the hypersphere of radius 1, of the function $V = \frac{1}{r^p} P$,

$$\Delta_1 V = p^2 (1 - V^2), \quad \Delta_2 V = -p\ (p + n - 1) V;$$

the hypersurfaces $V = c^{\text{te}}$ thus form an isoparametric system.

Moreover we can see directly that the function V varies between -1 and $+1$, because at any maximum or minimum of this function we have $\frac{\partial P}{\partial x_i} = \lambda x_i$, from which by elevation to the square and addition, $\lambda^2 = p^2,\ \lambda = \pm p$. By multiplying by x_i and summing, we find, at a maximum and at a minimum,

$$p\ P = \lambda, \quad \text{whence} \quad P = \pm 1.$$

35. The problem of finding families of isoparametric hypersurfaces, all of whose principal curvatures have the same degree of multiplicity is thus reduced to a purely algebraic problem, the one that we solved for $p = 3$ in the first part of this Memoir.

For $p = 4$, in the spherical space of 5 dimensions, there exists one and only one family of isoparametric hypersurfaces with 4 distinct principal curvatures, given by the equation

$$\begin{aligned} \cos 4t &= (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 \\ &- 2\ (x_3^2 - x_4^2 - 2x_1 x_5 + 2x_2 x_6)^2 - 2\ (2x_3 x_4 - 2x_1 x_6 - 2x_2 x_5)^2. \end{aligned}$$

(Received on 22. December 1938.)