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Families of isoparametric surfaces in spaces of constant curvature.

(Extract from a letter to M. Beniamino Segre)

by ÉLIE CARTAN (in Paris).

... The results you have obtained on isoparametric families of hypersurfaces of euclidean space of n dimensions[1] can be extended to n-dimensional spaces with constant negative curvature and, in part, to spaces with constant positive curvature. In both cases, it can be shown that the problem reduces to finding hypersurfaces whose principal curvatures are all constant, but whereas for spaces of negative or zero constant curvature, the number of distinct principal curvatures does not exceed two, this is no longer the case if the space has constant positive curvature. The problem in this last case is complicated, but very interesting; I was able to completely solve it for the space of constant positive curvature in 4 dimensions; moreover, in the

[1] Rendic. Lincei, V1 s., 27, 1938, p. 203–207.

general case, one can find a law giving the principal curvatures, as soon as one knows their degrees of multiplicity.

The method I employed uses a different technique than yours: it is the method of moving frames, related, as you know, to the method of normal congruences of G. RICCI.

1. Preliminaries. – I assume the fundamental form of the space, with n dimensions and constant riemannian curvature C, decomposed into a sum of n squares $\omega_1^2 + \cdots + \omega_n^2$, with the structural formulas

$$\begin{cases} \omega_i' = [\omega_k \ \omega_{ki}], \\ \omega_{ij}' = [\omega_{ik} \ \omega_{kj}] - C[\omega_i \ \omega_j], \end{cases} \tag{1}$$

where we have

$$\omega_{ij} = -\omega_{ji} = \gamma_{ijk} \ \omega_k; \tag{2}$$

the γ_{ijk} are the rotation coefficients of RICCI.

Let f be a function enjoying the property that its two differential parameters $\Delta_1 f$ and $\Delta_2 f$ are functions of f. I will assume that the n^{th} vector of the rectangular frame attached to the decomposition into squares is normal to the level hypersurfaces of the function f; the differential df is then a multiple $\varphi\omega_n$ of ω_n; φ is precisely the square root of $\Delta_1 f$, hence a function of f, so that ω_n is an exact differential dt. Hence the well-known result that the level hypersurfaces of the function f form a parallel family of hypersurfaces, and this family is known as soon as one knows one hypersurface of it.

From the property that the form ω_n is an exact differential, it follows that $\omega_n' = 0$, and hence by (1),

$$[\omega_i \ \omega_{in}] = 0; \tag{3}$$

this means that the ω_{in} are linear combinations of $\omega_1, \omega_2, \ldots, \omega_{n-1}$ with a symmetric array of coefficients.

Now the quadratic form $\omega_i \ \omega_{in}$, which is the second fundamental form of the hypersurface $t = C^{te}$, can be reduced to a sum of squares by a suitable choice of the first $n-1$ vectors of the frame; we can therefore write the formula,

$$\omega_{in} = a_i \omega_i \quad \text{(do not sum)}, \tag{4}$$

in which the a_i are the $n-1$ principal curvatures of the hypersurface.

We now introduce the differential parameter $\Delta_2 f$. The covariant derivatives f_i of the function f *with respect to the chosen frame* are

$$f_1 = 0,\ f_2 = 0, \ldots, f_{n-1} = 0,\ f_n = \varphi;$$

we have on the other hand, denoting by D the symbol of the covariant differentiation,

$$Df_i = df_i - f_k\omega_{ik} = df_i - f_k\gamma_{ikh}\omega_h = \begin{cases} -\varphi\gamma_{inh}\omega_h = -\varphi a_i\omega_i & \text{if } i = 1, \ldots, n-1; \\ d\varphi = \psi\omega_n & \text{if } i = n; \end{cases}$$

ψ is obviously a function of f. So we have

$$\Delta_2 f = f_{11} + f_{22} + \cdots + f_{nn} = \psi - (a_1 + a_2 + \cdots a_{n-1})\varphi.$$

We thus arrive at the following conclusion.

The necessary and sufficient condition for $\Delta_2 f$ to be a function of f, assuming that $\Delta_1 f$ is already a function of f, is that the level hypersurfaces of the function f all have constant mean curvature.

2. The principal curvatures of the level hypersurfaces. – The exterior derivative of equation (4) gives, taking into account (1),

$$(a_k - a_i)[\omega_k\ \omega_{ki}] + [\omega_i(da_i - \overline{C + a_i^2}\ \omega_n)] = 0, \tag{5}$$

the summation index k varying only from 1 to $n-1$, as in the rest of the formulas which follow. Since the first sum of the first member does not contain any term in $[\omega_i\ \omega_n]$, because the form ω_{ii} is identically zero, we have

$$da_i = (C + a_i^2)\ \omega_n + \cdots, \tag{6}$$

the terms not indicated depend linearly on $\omega_1, \omega_2, \ldots, \omega_{n-1}$. But since $da_1 + da_2 + \cdots + da_{n-1}$ only depends on ω_n, we deduce

$$d(a_1 + a_2 + \cdots + a_{n-1}) = [(n-1)C + a_1^2 + a_2^2 + \cdots + a_{n-1}^2]\ \omega_n;$$

hence $a_1^2 + a_2^2 + \cdots + a_{n-1}^2$ is also a function of f. The consideration of its differential shows, taking account of (6) and by a calculation analogous to the preceding one, that $a_1^3 + a_2^3 + \cdots + a_{n-1}^3$ is also a function of f, and so on. From which follows the theorem:

The level hypersurfaces of f have all of their principal curvatures constant. We also have

$$da_i = (C + a_i^2)\ \omega_n = (C + a_i^2)\ dt. \tag{7}$$

Reciprocally, let Σ be a particular hypersurface with constant principal curvatures; the parallel hypersurfaces form an isoparametric family. It will suffice to show this in a space of constant curvature equal to 1. If we set $a_i = \tan\theta_i$, the corresponding principal curvature of the hypersurface parallel to Σ at distance t is $\tan(\theta_i + t)$; it is therefore constant on that hypersurface.

3. A fundamental formula. – We return to equation (5) and set the collection of terms in $[\omega_k\ \omega_h]$ equal to zero; we obtain

$$(a_k - a_i)\gamma_{kih} = (a_h - a_i)\gamma_{hik} \quad \text{(do not sum)},$$

a formula from which we draw two conclusions:

1° the coefficient γ_{kih} is zero if $a_i = a_h \neq a_k$:

$$\gamma_{kih} = 0 \text{ if } a_i = a_h \neq a_k \text{ or } a_k = a_h \neq a_i. \tag{8}$$

2° We have, for a_i, a_k, a_h, distinct,

$$(a_k - a_i)\gamma_{kih} = (a_i - a_h)\gamma_{ihk} = (a_h - a_k)\gamma_{hki} \quad (a_i, a_k, a_h \text{ distinct}), \tag{9}$$

which allows us to write

$$\gamma_{ikh} = \frac{\lambda_{ikh}}{a_i - a_k}, \tag{10}$$

the quantity λ_{ikh} being symmetric with respect to its three indices.[2]

Given this we compute the exterior derivative of equation (2),

$$\omega_{ij} = \gamma_{ijk}\omega_k, \tag{2}$$

where we make the essential assumption that $a_i \neq a_j$, and equate the set of terms in $[\omega_i\ \omega_j]$ on the two sides. Taking into account equations (1), in which the index of summation can take the value n, we obtain

$$\sum_r(\gamma_{iri}\gamma_{rjj} - \gamma_{irj}\gamma_{rji}) - (C + a_i a_j) = \sum_r \gamma_{ijr}(\gamma_{irj} - \gamma_{jri}),$$

[2]We also have $\gamma_{kin} = 0$ if $a_k \neq a_i$. This relation will be used in n^{o} 4.

or again

$$C + a_i a_j = \sum_r (\gamma_{iri}\gamma_{rjj} + \gamma_{ijr}\gamma_{jri} + \gamma_{rij}\gamma_{ijr} + \gamma_{irj}\gamma_{jri}),$$

a formula in which the index of summation r varies from 1 to $n-1$.

If in the second member, we give r a fixed value such that $a_r = a_i$, the coefficients $\gamma_{rjj}, \gamma_{ijr}, \gamma_{jri}$ are zero, taking into account (8); it is the same if $a_r = a_j$; it is therefore sufficient to give r the values for which a_r is different from a_i and from a_j. Taking into account (10), we then find the *fundamental formula*

$$C + a_i a_j = 2 \sum_r \frac{\lambda_{ijr}^2}{(a_i - a_r)(a_j - a_r)}, \tag{11}$$

the summation being carried out for indices r for which $a_r \neq a_i, a_j$.

4. Case where the hypersurface has only two distinct principal curvatures. – In this case, the second member of equation (11) is zero, and we see that *the product of the two principal curvatures of the hypersurface is equal to $-C$, the negative of the riemannian curvature of the ambient space.*

We change notation and designate by Latin letters $i, j, \ldots,$ the indices corresponding to $a_i = a$ and by Greek letters $\alpha, \beta, \ldots,$ the indices corresponding to $a_\alpha = b$ $(ab = -C)$. By (8), all the coefficients $\gamma_{i\alpha k}, \gamma_{i\alpha\beta}$ and $\gamma_{i\alpha n}$ are zero; thus, we have

$$\omega_{i\alpha} = 0.$$

The formulas (1) first give

$$\left\{ \begin{array}{l} \omega_i' = [\omega_k\ \omega_{ki}] + a\ [\omega_i\ \omega_n], \\ \omega_{ij}' = [\omega_{ik}\ \omega_{kj}] - (C + a^2)[\omega_i\ \omega_j], \end{array} \right. \tag{12}$$

then

$$\left\{ \begin{array}{l} \omega_\alpha' = [\omega_\beta\ \omega_{\beta\alpha}] + b\ [\omega_\alpha\ \omega_n], \\ \omega_{\alpha\beta}' = [\omega_{\alpha\gamma}\ \omega_{\gamma\beta}] - (C + b^2)[\omega_\alpha\ \omega_\beta]. \end{array} \right. \tag{13}$$

Let us first suppose that the curvature of the space is positive and equal to 1; one can suppose, according to (7),

$$a = \tan t, \quad b = -\cot t.$$

We set

$$\omega_i = \cos t\ \widetilde{\omega}_i, \quad \omega_\alpha = \sin t\ \widetilde{\omega}_\alpha,$$

from which

$$\omega_{in} = \sin t\ \widetilde{\omega}_i, \quad \omega_{\alpha n} = -\cos t\ \widetilde{\omega}_\alpha.$$

Substituting into (12) and (13), we get

$$\begin{cases} \widetilde{\omega}'_i = [\widetilde{\omega}_k\ \omega_{ki}], \\ \omega'_{ij} = [\omega_{ik}\ \omega_{kj}] - [\widetilde{\omega}_i\ \widetilde{\omega}_j], \end{cases} \tag{12'}$$

and

$$\begin{cases} \widetilde{\omega}'_\alpha = [\widetilde{\omega}_\beta\ \omega_{\beta\alpha}], \\ \omega'_{\alpha\beta} = [\omega_{\alpha\gamma}\ \omega_{\gamma\beta}] - [\widetilde{\omega}_\alpha\ \widetilde{\omega}_\beta]. \end{cases} \tag{13'}$$

These formulas show that the two differential forms $\sum \widetilde{\omega}_i^2$ and $\sum \widetilde{\omega}_\alpha^2$ are of constant curvature equal to 1. Suppose the first form has p variables and the second has q variables ($p + q = n - 1$). We can realize the first by means of $p + 1$ variables $u_1, u_2, \ldots, u_{p+1}$ whose sum of squares is equal to 1, and the second by means of $q + 1$ new variables $u_{p+2}, u_{p+3}, \ldots, u_{n+1}$ whose sum of squares is also equal to 1. We will then have

$$\begin{aligned} \widetilde{\omega}_1^2 + \widetilde{\omega}_2^2 + \cdots + \widetilde{\omega}_p^2 &= du_1^2 + du_2^2 + \cdots + du_{p+1}^2, \\ \widetilde{\omega}_{p+1}^2 + \widetilde{\omega}_{p+2}^2 + \cdots + \widetilde{\omega}_{n-1}^2 &= du_{p+2}^2 + du_{p+3}^2 + \cdots + du_{n+1}^2, \end{aligned}$$

and hence, for the ds^2 of the space,

$$ds^2 = \cos^2 t\ (du_1^2 + \cdots + du_{p+1}^2) + \sin^2 t\ (du_{p+2}^2 + \cdots + du_{n+1}^2) + dt^2.$$

Finally, we set

$$x_i = \cos t\ u_i\ (i = 1, 2, \ldots, p+1), \quad x_\alpha = \sin t\ u_\alpha\ (\alpha = p+2, \ldots, n+1);$$

we immediately see that we have

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2,$$

the sum of the squares of the $n + 1$ variables being equal to 1. We find the ds^2 of the hypersphere of radius 1 in the euclidean space of $n+1$ dimensions, that is, the ds^2 of the spherical space of n dimensions, and we see that the isoparametric hypersurfaces are given by the equation

$$x_{p+2}^2 + \cdots + x_{n+1}^2 = \tan^2 t\ (x_1^2 + \cdots + x_{p+1}^2), \tag{14}$$

or again

$$\cos 2t = x_1^2 + \cdots + x_{p+1}^2 - (x_{p+2}^2 + \cdots + x_{n+1}^2). \qquad (14')$$

Among these hypersurfaces, two are singular, they are those which correspond to $t = 0$ and $t = \frac{\pi}{2}$; the first gives

$$x_{p+2} = \cdots = x_{n+1} = 0;$$

the second gives

$$x_1 = \cdots = x_{p+1} = 0:$$

these are two completely orthogonal and complementary totally geodesic varieties.

Now suppose that the curvature of the space is negative and equal to -1. We can set

$$a = -\tanh t, \quad b = -\coth t,$$

in such a way as to respect the relation (7). We further set

$$\begin{aligned} \omega_i &= \cosh t\, \widetilde{\omega}_i, & \omega_\alpha &= \sinh t\, \widetilde{\omega}_\alpha, \\ \omega_{in} &= -\sinh t\, \widetilde{\omega}_i, & \omega_{\alpha n} &= -\cosh t\, \widetilde{\omega}_\alpha. \end{aligned}$$

Formulas (12) and (13) now take the form

$$\begin{cases} \widetilde{\omega}'_i = [\widetilde{\omega}_k\ \omega_{ki}], \\ \omega'_{ij} = [\omega_{ik}\ \omega_{kj}] + [\widetilde{\omega}_i\ \widetilde{\omega}_j]; \end{cases} \qquad (12'')$$

$$\begin{cases} \widetilde{\omega}'_\alpha = [\widetilde{\omega}_\beta\ \omega_{\beta\alpha}], \\ \omega'_{\alpha\beta} = [\omega_{\alpha\gamma}\ \omega_{\gamma\beta}] - [\widetilde{\omega}_\alpha\ \widetilde{\omega}_\beta]. \end{cases} \qquad (13'')$$

The two differential forms $\sum \widetilde{\omega}_i^2$ and $\sum \widetilde{\omega}_\alpha^2$ are such that the first has constant curvature -1, and the second has constant curvature $+1$. We will realize them by setting

$$\sum \widetilde{\omega}_i^2 = du_1^2 + \cdots + du_p^2 - du_{n+1}^2, \quad \sum \widetilde{\omega}_\alpha^2 = du_{p+1}^2 + \cdots + du_n^2,$$

with $n+1$ variables linked by the relations

$$u_{n+1}^2 - u_1^2 - \cdots - u_p^2 = 1, \quad u_{p+1}^2 + \cdots + u_n^2 = 1.$$

We then will have for the ds^2 of the space

$$ds^2 = \cosh^2 t\ (du_1^2 + \cdots + du_p^2 - du_{n+1}^2) + \sinh^2 t\ (du_{p+1}^2 + \cdots + du_n^2) + dt^2.$$

It will suffice to set

$$x_i = \cosh t \ u_i \ (i = 1, \dots, p, n+1), \quad x_\alpha = \sinh t \ u_\alpha \ (\alpha = p+1, \dots, n),$$

and we will have

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2,$$

with

$$x_{n+1}^2 - x_1^2 - x_2^2 - \cdots - x_n^2 = 1.$$

We have found the CAYLEY-KLEIN representation with the *absolute*

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = 0.$$

The level hypersurfaces are given by the equation

$$x_{p+1}^2 + \cdots + x_n^2 = \tanh^2 t \ (x_{n+1}^2 - x_1^2 - \cdots - x_p^2), \tag{15}$$

which can also be written in the form

$$\cosh 2t = x_{n+1}^2 - x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2, \tag{15'}$$

the second member being a function f whose first two differential parameters are functions of f.

For $t = 0$ we have a singular hypersurface with equations

$$x_{p+1} = x_{p+2} = \cdots = x_n = 0;$$

it is a totally geodesic linear variety of p dimensions.

5. Case where the principal curvatures are all equal to each other. – If $C = 1$, we can set the principal curvature $a = \tan t$ and use the formulas (14) assuming that $p = n - 1$: we obtain the hyperspheres of fixed center

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \cot^2 t \ x_{n+1}^2;$$

for $t = \pi/2$, we find the point $x_1 = \cdots = x_n = 0$, the common center of the hyperspheres.

If $C = -1$, we find by setting either $a = -\tanh t$, or $a = \coth t$, the hypersurfaces obtained by giving to p in (15) the value $n - 1$ and the value 0, namely

$$x_n^2 = \tanh^2 t \ (x_{n+1}^2 - x_1^2 - \cdots - x_{n-1}^2), \tag{16}$$

and

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \tanh^2 t \; x_{n+1}^2. \tag{17}$$

The hypersurfaces (17) are the hyperspheres with center $x_1 = x_2 = \cdots = x_n = 0$; the hypersurfaces (16) are the *equidistant* hypersurfaces of the hyperplane $x_n = 0$.

But there is still another solution which does not follow from the results of n$^{\text{o}}$ 4, it is the one that corresponds to $a = 1$ (the case $a = -1$ is not distinct from it). By setting

$$\omega_i = e^{-t}\widetilde{\omega}_i, \quad \omega_{in} = e^{-t}\widetilde{\omega}_i,$$

the equations (1) become

$$\begin{cases} \widetilde{\omega}_i' = [\omega_{ik} \; \widetilde{\omega}_k], \\ \omega_{ij}' = [\omega_{ik} \; \omega_{kj}]; \end{cases} \tag{18}$$

they prove that the form $\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2 \cdots + \widetilde{\omega}_{n-1}^2$ has zero curvature; we can therefore set

$$ds^2 = e^{-2t}(du_1^2 + du_2^2 + \cdots + du_{n-1}^2) + dt^2,$$

or again, by setting $e^t = u_n$,

$$ds^2 = \frac{du_1^2 + du_2^2 + \cdots + du_n^2}{u_n^2}.$$

The hypersurfaces $u_n = C^{te}$ are *horospheres.* In the representation of CAYLEY-KLEIN, the absolute being the quadric

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = 0,$$

the corresponding family of isoparametric hypersurfaces can be written

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 + e^{2t}(x_{n+1} - x_n)^2 = 0. \tag{19}$$

6. General results on the case where there are more than two distinct principal curvatures. – Recall the fundamental formula (11)

$$C + a_i a_j = 2 \sum_r \frac{\lambda_{ijr}^2}{(a_i - a_r)(a_j - a_r)}. \tag{11}$$

We will deduce that *the case where the hypersurfaces admit more than two distinct principal curvatures can only arise in a space of positive curvature.*

Suppose there are p distinct principal curvatures, for example $a_1, a_2, \ldots, a_p$ and let $\nu_1, \nu_2, \ldots, \nu_p$ be their respective degrees of multiplicity ($\nu_1 + \nu_2 + \cdots + \nu_p = n - 1$). Let

$$\rho_{ijk} = \frac{2 \sum \lambda_{\alpha\beta\gamma}^2}{(a_i - a_k)(a_j - a_k)(a_i - a_j)} \quad (i, j, k = 1, 2, \ldots, p;\ i \neq j \neq k), \tag{20}$$

the sum being extended to all the indices $\alpha = 1, 2, \ldots, n - 1$ for which $a_\alpha = a_i$, to all the indices $\beta = 1, 2, \ldots, n - 1$ for which $a_\beta = a_j$, to all the indices $\gamma = 1, 2, \ldots, n - 1$ for which $a_\gamma = a_k$. The ρ_{ijk} are obviously skew symmetric with respect to their three indices. We will also set

$$\rho_{ij} = -\rho_{ji} = \sum_k^{k \neq i,j} \rho_{ijk};$$

we obviously have

$$\sum_j^{i \neq j} \rho_{ij} = 0. \tag{21}$$

Given this, the equation (11) can be written

$$\nu_i \nu_j (C + a_i a_j) = \rho_{ij}(a_i - a_j). \tag{22}$$

If we regard ρ_{ij} as given, we obtain a homographic relation between a_i and a_j, and all these homographic relations have the same united points $\pm\sqrt{-C}$. This results a priori in the existence of a relation between $\rho_{ij}, \rho_{jk}, \rho_{ki}$. By effectively eliminating a_k between the equations (ik) and (jk) and comparing to equation (ij), we find

$$\nu_i^2 \nu_j^2 \nu_k^2 C = \nu_j \nu_k \rho_{ki} \rho_{ij} + \nu_k \nu_i \rho_{ij} \rho_{jk} + \nu_i \nu_j \rho_{jk} \rho_{ki}. \tag{23}$$

This relation will provide us with the announced theorem.

Note first that the quantities ρ_{ij} cannot all be zero, otherwise we would indeed have $a_i a_j = -C$; there exists at least one nonzero curvature a_i, and we would then have $a_i(a_j - a_k) = 0$, hence $a_j = a_k$, case excluded. So suppose that $\rho_{12} > 0$. The relation (21) for $i = 2$, combined with the inequality $\rho_{21} < 0$, shows that one of the quantities $\rho_{23}, \ldots, \rho_{2p}$ is positive, so take $\rho_{23} > 0$. Similarly, *if we assume that* $\rho_{31} < 0$, we can suppose that $\rho_{34} > 0$, and so on. There will necessarily come a time when we will have the sequence of inequalities $\rho_{12} > 0, \rho_{23} > 0, \ldots, \rho_{q-1,q} > 0$, with $\rho_{qi} > 0$ for an index $i < q-1$, none of the quantities $\rho_{q-1,k}$ being positive for $k < q-1$. But then the formula (23) for the indices $i, q-1, q$ shows immediately that C is positive. C. Q. F. D.

We have therefore completely solved the problem for C negative, and the solutions are given by the formulas (15) and (19). But it is not completely solved for C positive.

7. We can however give the way to find the law which gives the principal curvatures of isoparametric hypersurfaces as a function of t, knowing the number p of distinct principal curvatures and their degrees of multiplicity $\nu_1, \nu_2, \ldots, \nu_p$. Assuming $C = 1$, we set, according to (7),

$$a_i = \tan t_i,$$

where t_i only differs from t by a constant. The equation (22) can then be written

$$\rho_{ij} = \nu_i \nu_j \cot(t_i - t_j), \tag{22'}$$

and the relations (21) become

$$\sum_{j}^{j \neq i} \nu_i \nu_j \cot(t_i - t_j) = 0. \tag{21'}$$

The problem consists in solving these p equations, which further reduce to $p-1$, with the $p-1$ unknowns $t_i - t_j$.

Let us represent in a plane the p straight lines Δ_i with angular coefficients a_i passing through a fixed point O, and give ourselves the order in which they succeed each other when we turn around O in a determined direction. Suppose that we successively meet the straight lines $\Delta_1, \Delta_2, \ldots, \Delta_p$: we will have the inequalities

$$0 < t_2 - t_1 < t_3 - t_1 < \cdots < t_p - t_1 < \pi. \tag{24}$$

Consider p real variables $u_1, u_2, \dots, u_p$ subject to satisfying the same inequalities

$$0 < u_2 - u_1 < u_3 - u_1 < \cdots < u_p - u_1 < \pi,$$

and form the essentially positive function

$$F(u) = \prod_{i<j}^{1,2,\dots,p} [\sin(u_j - u_i)]^{\nu_i \nu_j};$$

the equations (21′) express that all first-order partial derivatives of this function vanish for $u_i = t_i$. It is easy to see that this function F attains its absolute maximum for these values (assumed to exist). Indeed the TAYLOR formula gives

$$F(u) - F(t) = \frac{1}{2}(u_i - t_i)(u_j - t_j)\frac{\partial^2 F(v)}{\partial v_i \partial v_j}, \text{ with } v_i = t_i + \theta(u_i - t_i) \ (0 < \theta < 1).$$

But the calculation of the partial derivatives of F leads for the second member to the value

$$-\frac{1}{2}\sum_{i<j} \frac{\nu_i \nu_j [(u_i - t_i) - (u_j - t_j)]^2}{\sin^2(v_i - v_j)}.$$

We therefore have $F(u) < F(v)$, unless $u_i - t_i = u_j - t_j$, that is, unless we increase the t_i by the same arbitrary constant.

It follows from this that the system (18′) can admit only one solution for the differences $t_i - t_j$, if we suppose the t_i to be subject to the inequalities (21). But the function $F(u)$ obviously admits an absolute maximum when we subject the u_i to these inequalities; *thus there is a unique law which gives the principal curvatures of the isoparametric hypersurfaces once we give ourselves the order in which the lines follow one another* $\Delta_1, \Delta_2, \dots, \Delta_p$. It is clear that if all the principal curvatures are simple, the consecutive angles t_i differ by $\frac{\pi}{n-1}$, and we can assume

$$a_i = \tan\left(t + \frac{i\pi}{n-1}\right), \quad (i = 1, 2, \dots, n-1).$$

If we simply give ourselves the ν_i, there will be as many solutions as there are different orders for the straight lines, two orders not being regarded as different if, by writing next to each line Δ_i the corresponding integer ν_i, the two figures considered present the same sequence of integers in the same order, when either we follow the lines both times in the same direction of rotation, or we follow them the first time in one direction, and the second time in the opposite direction. If all the integers ν_i are equal, there is obviously only one solution, the one indicated above; for $p = 4$, there are 3 solutions if the 4 indices of multiplicity ν_i are distinct, two solutions if the number of distinct indices of multiplicity is 3 or 2, one solution if all the indices of multiplicity are equal to each other.

8. The case of the spherical space of four dimensions. – It is not obvious a priori that there always exists a family of isoparametric hypersurfaces corresponding to one of the laws which have just been indicated. It can easily be shown that this is so for $n = 4$; at the same time, this will completely solve the problem of families of isoparametric hypersurfaces in 4-dimensional spherical space.

We have here $\nu_1 = \nu_2 = \nu_3 = 1$, and we can assume

$$a_1 = \tan\left(t - \frac{\pi}{3}\right), \quad a_2 = \tan t, \quad a_3 = \tan\left(t + \frac{\pi}{3}\right).$$

The formulas (22′) give

$$\rho_{23} = \rho_{31} = \rho_{12} = -\cot\frac{\pi}{3} = -\frac{1}{\sqrt{3}} = \rho_{123},$$

whence, according to (20),

$$\lambda_{123}^2 = -\frac{1}{2\sqrt{3}}(a_1 - a_2)(a_1 - a_3)(a_2 - a_3) =$$

$$\frac{1}{2\sqrt{3}}\frac{\sin(t_1 - t_2)\sin(t_2 - t_3)\sin(t_3 - t_1)}{\cos^2 t_1 \cos^2 t_2 \cos^2 t_3} = \frac{3}{16\cos^2 t_1 \cos^2 t_2 \cos^2 t_3}.$$

We can take

$$\lambda_{123}\cos t_1 \cos t_2 \cos t_3 = \frac{\sqrt{3}}{4}.$$

We then have, according to (2) and (10),

$$\begin{aligned}
\omega_{23} &= \frac{\lambda_{123}}{a_2 - a_3}\omega_1 = \frac{\lambda_{123}\cos t_2 \cos t_3}{\sin(t_2 - t_3)}\omega_1 = -\frac{1}{2}\frac{\omega_1}{\cos t_1},\\
\omega_{31} &= \frac{\lambda_{123}}{a_3 - a_1}\omega_2 = \frac{\lambda_{123}\cos t_3 \cos t_1}{\sin(t_3 - t_1)}\omega_2 = \frac{1}{2}\frac{\omega_2}{\cos t_2},\\
\omega_{12} &= \frac{\lambda_{123}}{a_1 - a_2}\omega_3 = \frac{\lambda_{123}\cos t_1 \cos t_2}{\sin(t_1 - t_2)}\omega_3 = -\frac{1}{\sqrt{2}}\frac{\omega_3}{\cos t_3}.
\end{aligned}$$

It is therefore natural to set

$$\omega_1 = 2\,\widetilde{\omega}_1 \cos t_1, \quad \omega_2 = 2\,\widetilde{\omega}_2 \cos t_2, \quad \omega_3 = 2\,\widetilde{\omega}_3 \cos t_3, \tag{25}$$

whence

$$\omega_{14} = 2\,\widetilde{\omega}_1 \sin t_1, \quad \omega_{24} = 2\,\widetilde{\omega}_2 \sin t_2, \quad \omega_{34} = 2\,\widetilde{\omega}_3 \sin t_3, \tag{26}$$

with

$$\omega_{23} = -\widetilde{\omega}_1, \quad \omega_{31} = \widetilde{\omega}_2, \quad \omega_{12} = -\widetilde{\omega}_3. \tag{27}$$

The structural formulas (1) give, after simplifications,

$$\widetilde{\omega}_1' = [\widetilde{\omega}_2\ \widetilde{\omega}_3], \quad \widetilde{\omega}_2' = [\widetilde{\omega}_3\ \widetilde{\omega}_1], \quad \widetilde{\omega}_3' = [\widetilde{\omega}_1\ \widetilde{\omega}_2]; \tag{28}$$

the quadratic differential form $\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2 + \widetilde{\omega}_3^2$ has curvature 1.

We can arrive at the desired hypersurfaces by searching for a hypersurface for which one of the $\cos t_i$ is zero, for example $\cos t_3$, because then the ds^2

$$ds^2 = 4\cos^2 t_1\ \widetilde{\omega}_1^2 + 4\cos^2 t_2\ \widetilde{\omega}_2^2 + 4\cos^2 t_3\ \widetilde{\omega}_3^2$$

will reduce to a sum of two squares, namely

$$ds^2 = 3(\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2).$$

Moreover, for the surface thus considered, we will have

$$\left\{\begin{array}{ll} \omega_1 = \sqrt{3}\ \widetilde{\omega}_1, & \omega_2 = \sqrt{3}\ \widetilde{\omega}_2 \\ \omega_3 = 0, & \omega_4 = 0, \\ \omega_{13} = -\widetilde{\omega}_2, & \omega_{23} = -\widetilde{\omega}_1, \\ \omega_{14} = \widetilde{\omega}_1, & \omega_{24} = \widetilde{\omega}_2. \end{array}\right. \tag{29}$$

The two asymptotic forms of this surface are

$$\begin{array}{rcl} \omega_1\omega_{13} + \omega_2\omega_{23} & = & -2\sqrt{3}\ \widetilde{\omega}_1\widetilde{\omega}_2, \\ \omega_1\omega_{14} + \omega_2\omega_{24} & = & \sqrt{3}\ (\widetilde{\omega}_2^2 - \widetilde{\omega}_1^2), \end{array}$$

so that the normal curvature of any curve drawn on the surface is

$$\frac{\sqrt{3}\sqrt{(\widetilde{\omega}_1^2 - \widetilde{\omega}_2^2)^2 + 4\ \widetilde{\omega}_1^2\ \widetilde{\omega}_2^2}}{3(\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2)} = \frac{1}{\sqrt{3}};$$

it is constant. Now M. BORUVKA has demonstrated[3] that all surfaces of the elliptical space of 4 dimensions on which all curves have the same constant normal curvature are the representative surfaces of the harmonic polynomials of the second degree in three variables and are applicable on the sphere, or rather on the elliptical plane. In the spherical space of 4 dimensions, we can represent the surface considered by the following formulas with respect to the coordinates x_1, x_2, x_3, x_4, x_5, whose sum of squares is equal to 1,

$$\begin{gathered} x_1 = \sqrt{3}vw, \quad x_2 = \sqrt{3}wu, \quad x_3 = \sqrt{3}uv, \\ x_4 = \sqrt{3}\ \frac{u^2 - v^2}{2}, \quad x_5 = w^2 - \frac{u^2 + v^2}{2}, \end{gathered} \tag{30}$$

where u, v, w are three parameters linked by the relation

$$u^2 + v^2 + w^2 = 1. \tag{31}$$

The parallel hypersurfaces will be obtained by looking for the envelope, in the spherical space of 4 dimensions, of the hyperspheres of radius t having their centers at the various points of the surface. These hyperspheres have the general equation

$$\sqrt{3}vwx_1 + \sqrt{3}wux_2 + \sqrt{3}uvx_3 + \sqrt{3}\ \frac{u^2 - v^2}{2}x_4 + \left(w^2 - \frac{u^2 + v^2}{2}\right) x_5 = \cos t. \tag{32}$$

Their envelope is provided by the relations

$$\begin{aligned} \sqrt{3}\ wx_2 + \sqrt{3}\ vx_3 + \sqrt{3}\ ux_4 - ux_5 &= \lambda u, \\ \sqrt{3}\ wx_1 + \sqrt{3}\ ux_3 - \sqrt{3}\ vx_4 - vx_5 &= \lambda v, \\ \sqrt{3}\ vx_1 + \sqrt{3}\ ux_2 + 2wx_5 &= \lambda w; \end{aligned}$$

taking into account (32), we find $\lambda = 2\cos t$. Then the envelope has the equation

$$\begin{vmatrix} \sqrt{3}\ x_4 - x_5 - 2\cos t & \sqrt{3}\ x_3 & \sqrt{3}\ x_2 \\ \sqrt{3}\ x_3 & -\sqrt{3}\ x_4 - x_5 - 2\cos t & \sqrt{3}\ x_1 \\ \sqrt{3}\ x_2 & \sqrt{3}\ x_1 & 2x_5 - 2\cos t \end{vmatrix} = 0,$$

or again, by developing,

$$\cos 3t = x_5^3 + \frac{3}{2}(x_1^2 + x_2^2)x_5 - 3(x_3^2 + x_4^2)x_5 + \frac{3\sqrt{3}}{2}(x_1^2 - x_2^2)x_4 + 3\sqrt{3}\ x_1x_2x_3. \tag{33}$$

[3] Comptes rendus, 187, pp. 334–336, 1928.

This is the equation of the isoparametric hypersurfaces that we have been seeking. We see that if we are in *spherical* space, we obtain the same hypersurface when we increase t by $2\pi/3$; if we are in *elliptical* space, it suffices to increase t by $\pi/3$. In this last case, there is only one singular hypersurface, it is the surface (31). In the first case, there are two, namely the surface (31) and it antipode. The equations of these surfaces can be put in the form

$$\frac{\partial P}{\partial x_i} \mp 3x_i = 0 \quad (i = 1, 2, \ldots, 5),$$

denoting by P the polynomial which is the second member of (33).

One can verify a posteriori in a simple way the isoparametrism of the hypersurfaces (33). For the polynomial P to enjoy, in the spherical space of four dimensions, the property that $\Delta_1 P$ and $\Delta_2 P$ are functions of P, it suffices that the two differential parameters $\Delta_1 P$ and $\Delta_2 P$, *calculated in the euclidean space of 5 dimensions*, become functions of P on the hypersphere of radius 1. But we easily find

$$\sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 = 9\ (x_1^2 + x_2^2 + \cdots + x_5^2)^2 = 9, \quad \sum_i \frac{\partial^2 P}{\partial x_i^2} = 0;$$

the verification is done. This remark suggests the problem of finding, in n-dimensional euclidean space, all the homogeneous polynomials P such that $\Delta_1 P$ and $\Delta_2 P$ are, up to constant factors, powers of $x_1^2 + x_2^2 + \cdots + x_n^2$; if n is odd, this requires $\Delta_2 P$ to be zero. The polynomials which are presented in N° 4 (formula 14′), obviously enjoy the two properties in question.

I will add finally that the hypersurfaces (33) admit a three-parameter transitive group of displacements of the ambient space. It would be interesting to know if every hypersurface with constant principal curvatures in a space of constant positive curvature admits a transitive group of rigid displacements; this is so if the number of distinct principal curvatures is at most two, but it is not obvious otherwise[4].

[4]During the printing, I was able to determine all of the isoparametric families with three distinct principal curvatures; they only exist in spaces of 4, 7, 13, and 25 dimensions. They will be the subject of a future article.