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Classifications of Dupin Hypersurfaces in Lie Sphere Geometry

Thomas E. Cecil

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Abstract

This is a survey of local and global classification results concerning Dupin hypersurfaces in S^n (or \mathbf{R}^n) that have been obtained in the context of Lie sphere geometry. The emphasis is on results that relate Dupin hypersurfaces to isoparametric hypersurfaces in spheres. Along with these classification results, many important concepts from Lie sphere geometry, such as curvature spheres, Lie curvatures, and Legendre lifts of submanifolds of S^n (or \mathbf{R}^n), are described in detail. The paper also contains several important constructions of Dupin hypersurfaces with certain special properties.

An oriented hypersurface $f : M^{n-1} \rightarrow S^n$ (or \mathbf{R}^n) is said to be *Dupin* if:

- (a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

- (b) the number g of distinct principal curvatures is constant on M .

An important class of proper Dupin hypersurfaces consists of the isoparametric (constant principal curvatures) hypersurfaces in S^n , and those hypersurfaces in \mathbf{R}^n obtained from isoparametric hypersurfaces in S^n via stereographic projection. For example, a standard product torus $S^1(r) \times S^1(s) \subset S^3$,

$r^2 + s^2 = 1$, is an isoparametric hypersurface with two principal curvatures, and the well-known ring cyclides of Dupin [32], [9, pp. 148–159] in \mathbf{R}^3 are obtained from standard product tori via stereographic projection.

In this paper, we will discuss various local and global classification results for proper Dupin hypersurfaces in S^n (or \mathbf{R}^n) that have been obtained in the context of Lie sphere geometry. The emphasis is on results that relate Dupin hypersurfaces to isoparametric hypersurfaces in spheres.

Along with these classification results, many important concepts from Lie sphere geometry, such as curvature spheres, Lie curvatures, and Legendre lifts of submanifolds of S^n , are described in detail. The paper also contains several important constructions of Dupin hypersurfaces with certain special properties. The presentation is based primarily on the author’s book [9], and some passages are direct quotations from that book.

1 Basic Concepts of Lie Sphere Geometry

1.1 Indefinite space forms and projective space

We begin with some preliminary remarks on indefinite scalar product spaces and projective geometry, along with the corresponding notation. A *scalar product* is a nondegenerate bilinear form on a real vector space V . We will use the notation \mathbf{R}_k^n for the vector space $V = \mathbf{R}^n$ endowed with a scalar product having signature $(n - k, k)$, and we will primarily consider the cases where $k = 0, 1$ or 2 . However, at times we will consider subspaces of \mathbf{R}_k^n on which the bilinear form is degenerate.

Let (x, y) be the indefinite scalar product of signature $(n, 1)$ on the Lorentz space \mathbf{R}_1^{n+1} defined by

$$(x, y) = -x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}, \quad (1)$$

where $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$. We will call this scalar product the *Lorentz metric*. A vector x is said to be *spacelike*, *timelike* or *lightlike*, respectively, depending on whether (x, x) is positive, negative or zero. We will use this terminology even when we are using a metric of different signature. In Lorentz space, the set of all lightlike vectors, given by the equation,

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2, \quad (2)$$

forms a cone of revolution, called the *light cone*. Timelike vectors are “inside the cone” and spacelike vectors are “outside the cone.”

If x is a nonzero vector in \mathbf{R}_1^{n+1} , let x^\perp denote the orthogonal complement of x with respect to the Lorentz metric. If x is timelike, then the metric restricts to a positive definite form on x^\perp , and x^\perp intersects the light cone only at the origin. If x is spacelike, then the metric has signature $(n-1, 1)$ on x^\perp , and x^\perp intersects the cone in a cone of one less dimension. If x is lightlike, then x^\perp is tangent to the cone along the line through the origin determined by x . The bilinear form has signature $(n-1, 0)$ on this n -dimensional plane.

Lie sphere geometry is defined in the context of real projective space \mathbf{P}^n , so we now recall some concepts from projective geometry. We define an equivalence relation on $\mathbf{R}^{n+1} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t . We denote the equivalence class determined by a vector x by $[x]$. Projective space \mathbf{P}^n is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in \mathbf{R}^{n+1} . The rectangular coordinates (x_1, \dots, x_{n+1}) are called *homogeneous coordinates* of the point $[x] \in \mathbf{P}^n$, and they are only determined up to a nonzero scalar multiple.

The affine space \mathbf{R}^n can be embedded in \mathbf{P}^n as the complement of the hyperplane $(x_1 = 0)$ at infinity by the map $\phi : \mathbf{R}^n \rightarrow \mathbf{P}^n$ given by $\phi(u) = [(1, u)]$. A scalar product on \mathbf{R}^{n+1} , such as the Lorentz metric, determines a polar relationship between points and hyperplanes in \mathbf{P}^n . We will also use the notation x^\perp to denote the polar hyperplane of $[x]$ in \mathbf{P}^n , and we will call $[x]$ the *pole* of x^\perp .

If x is a lightlike vector in $\mathbf{R}^{n+1} - \{0\}$, then $[x]$ can be represented by a vector of the form $(1, u)$ for $u \in \mathbf{R}^n$. Then the equation $(x, x) = 0$ for the light cone becomes $u \cdot u = 1$ (Euclidean dot product), i.e., the equation for the unit sphere in \mathbf{R}^n . Hence, the set of points in \mathbf{P}^n determined by lightlike vectors in $\mathbf{R}^{n+1} - \{0\}$ is naturally diffeomorphic to the sphere S^{n-1} .

1.2 Möbius geometry of unoriented spheres

In this section, we study the space of all (unoriented) hyperspheres in Euclidean n -dimensional space \mathbf{R}^n and in the unit sphere $S^n \subset \mathbf{R}^{n+1}$. These two spaces of spheres are closely related via stereographic projection, as we now recall. We will always assume that $n \geq 2$. (See [9, pp. 11–14] for more detail.)

We denote the Euclidean dot product of two vectors u and v in \mathbf{R}^n by

$u \cdot v$. We first consider stereographic projection $\sigma : \mathbf{R}^n \rightarrow S^n - \{P\}$, where S^n is the unit sphere in \mathbf{R}^{n+1} given by $y \cdot y = 1$, and $P = (-1, 0, \dots, 0)$ is the south pole of S^n . The well-known formula for $\sigma(u)$ is

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right).$$

We next embed \mathbf{R}^{n+1} into \mathbf{P}^{n+1} by the embedding ϕ mentioned in the preceding section. Thus, we have the map $\phi\sigma : \mathbf{R}^n \rightarrow \mathbf{P}^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right) \right] = \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u \right) \right]. \quad (3)$$

Let (z_1, \dots, z_{n+2}) be homogeneous coordinates on \mathbf{P}^{n+1} and $(,)$ the Lorentz metric on the space \mathbf{R}_1^{n+2} . Then $\phi\sigma(\mathbf{R}^n)$ is the set of points in \mathbf{P}^{n+1} lying on the n -sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \dots, 0)]$ corresponding to the south pole P . We will refer to the points in Σ other than $[(1, -1, 0, \dots, 0)]$ as *proper points*, and we will call Σ the *Möbius sphere* or *Möbius space*.

The basic framework for the Möbius geometry of unoriented spheres is as follows. Suppose that ξ is a spacelike vector in \mathbf{R}_1^{n+2} . Then the polar hyperplane ξ^\perp to $[\xi]$ in \mathbf{P}^{n+1} intersects the sphere Σ in an $(n - 1)$ -sphere S^{n-1} . This sphere S^{n-1} is the image under $\phi\sigma$ of an $(n - 1)$ -sphere in \mathbf{R}^n , unless it contains the improper point $[(1, -1, 0, \dots, 0)]$, in which case it is the image under $\phi\sigma$ of a hyperplane in \mathbf{R}^n . Hence, we have a bijective correspondence between the set of all spacelike points in \mathbf{P}^{n+1} and the set of all hyperspheres and hyperplanes in \mathbf{R}^n .

We now find specific formulas for this correspondence. Consider the sphere in \mathbf{R}^n with center p and radius $r > 0$ given by the equation

$$(u - p) \cdot (u - p) = r^2, \quad u \in \mathbf{R}^n. \quad (4)$$

We now translate this into an equation involving the Lorentz metric and the corresponding polarity relationship on \mathbf{P}^{n+1} . A direct calculation shows that equation (4) is equivalent to the equation

$$(\xi, \phi\sigma(u)) = 0, \quad (5)$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right), \quad (6)$$

and $\phi\sigma(u)$ is given by equation (3). Thus, the point u is on the sphere given by equation (4) if and only if $\phi\sigma(u)$ lies on the polar hyperplane of $[\xi]$. Note that the first two coordinates of ξ satisfy $\xi_1 + \xi_2 = 1$, and that $(\xi, \xi) = r^2$. Although ξ is only determined up to a nonzero scalar multiple, we can conclude that $\eta_1 + \eta_2$ is not zero for any $\eta \simeq \xi$.

Conversely, given a spacelike point $[z]$ with $z_1 + z_2$ nonzero, we determine the corresponding sphere in \mathbf{R}^n as follows. Let $\xi = z/(z_1 + z_2)$ so that $\xi_1 + \xi_2 = 1$. Then from equation (6), the center of the corresponding sphere is the point $p = (\xi_3, \dots, \xi_{n+2})$, and the radius is the square root of (ξ, ξ) .

We next suppose that η is a spacelike vector with $\eta_1 + \eta_2 = 0$. Then

$$(\eta, (1, -1, 0, \dots, 0)) = 0.$$

In that case, the improper point $\phi(P)$ lies on the polar hyperplane of $[\eta]$, and the point $[\eta]$ corresponds to a hyperplane in \mathbf{R}^n . Again we can find an explicit correspondence as follows.

Consider the hyperplane in \mathbf{R}^n given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{7}$$

A direct calculation shows that (7) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \text{ where } \eta = (h, -h, N). \tag{8}$$

So the hyperplane (7) is represented in the polarity relationship by $[\eta]$. Conversely, let z be a spacelike point with $z_1 + z_2 = 0$. Then $(z, z) = v \cdot v$, where $v = (z_3, \dots, z_{n+2})$. Let $\eta = z/|v|$. Then η has the form (8) and $[z]$ corresponds to the hyperplane (7). Thus we have explicit formulas for the bijective correspondence between the set of spacelike points in \mathbf{P}^{n+1} and the set of hyperspheres and hyperplanes in \mathbf{R}^n .

The fundamental invariant of Möbius geometry is the angle. The study of angles in our setting is quite natural, since orthogonality between spheres and planes in \mathbf{R}^n can be expressed in terms of the Lorentz metric. Specifically, let S_1 and S_2 denote the spheres in \mathbf{R}^n with respective centers p_1 and p_2 and respective radii r_1 and r_2 . By the Pythagorean Theorem, the two spheres intersect orthogonally if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{9}$$

If these spheres correspond by equation (6) to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, then a calculation shows that equation (9) is equivalent to the condition

$$(\xi_1, \xi_2) = 0. \quad (10)$$

A hyperplane π in \mathbf{R}^n is orthogonal to a hypersphere S precisely when π passes through the center of S . If S has center p and radius r , and π is given by the equation $u \cdot N = h$, then the condition for orthogonality is just $p \cdot N = h$. If S corresponds to $[\xi]$ as in (6) and π corresponds to $[\eta]$ as in (8), then this equation for orthogonality is equivalent to $(\xi, \eta) = 0$. Finally, if two planes π_1 and π_2 are represented by $[\eta_1]$ and $[\eta_2]$ as in (8), then the orthogonality condition $N_1 \cdot N_2 = 0$ is equivalent to the equation $(\eta_1, \eta_2) = 0$.

A *Möbius transformation* is a projective transformation of \mathbf{P}^{n+1} which preserves the condition $(\eta, \eta) = 0$. This implies (see, for example, [9, p. 26]) that a Möbius transformation also preserves the relationship $(\eta, \xi) = 0$, and it maps spacelike points to spacelike points. Thus it preserves orthogonality (and hence angles) between spheres and planes in \mathbf{R}^n . In fact, the group of Möbius transformations is isomorphic to $O(n+1, 1)/\{\pm I\}$, where $O(n+1, 1)$ is the group of orthogonal transformations of the Lorentz space \mathbf{R}_1^{n+2} (see, for example, [9, p. 27]).

Note that a Möbius transformation takes lightlike vectors to lightlike vectors, and so it induces a conformal diffeomorphism of the sphere Σ onto itself. It is well known that the group of conformal diffeomorphisms of the sphere is precisely the Möbius group.

1.3 Lie geometry of oriented hyperspheres in \mathbf{R}^n

We now turn to the construction of Lie's [37] geometry of oriented spheres in \mathbf{R}^n . (See [9, pp. 14–18] for more detail.) Let W^{n+1} be the set of vectors in \mathbf{R}_1^{n+2} satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in \mathbf{R}_1^{n+2} . If α is a spacelike point in \mathbf{P}^{n+1} , then there are precisely two vectors $\pm\zeta$ in W^{n+1} with $\alpha = [\zeta]$. These two vectors can be taken to correspond to the two orientations of the oriented sphere or plane represented by α in the following way. First, embed \mathbf{R}_1^{n+2} into \mathbf{P}^{n+2} by the embedding $z \mapsto [(z, 1)]$. If $\zeta \in W^{n+1}$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta, 1)]$ in \mathbf{P}^{n+2} lies on the quadric Q^{n+1} in \mathbf{P}^{n+2} given in homogeneous coordinates by the equation

$$\langle x, x \rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0. \quad (11)$$

The manifold Q^{n+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (11) is called the *Lie metric* or *Lie scalar product*. We will let $\{e_1, \dots, e_{n+3}\}$ denote the standard orthonormal basis for the scalar product space \mathbf{R}_2^{n+3} with metric $\langle \cdot, \cdot \rangle$. Here e_1 and e_{n+3} are timelike and the rest are spacelike.

We now define a correspondence between the points on Q^{n+1} and the set of oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$. Suppose that x is any point on the quadric with homogeneous coordinate $x_{n+3} \neq 0$. Then x can be represented by a vector of the form $(\zeta, 1)$, where the Lorentz scalar product $(\zeta, \zeta) = 1$. Suppose first that $\zeta_1 + \zeta_2 \neq 0$. Then in Möbius geometry $[\zeta]$ represents a sphere in \mathbf{R}^n . If as in equation (6), we represent $[\zeta]$ by a vector of the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right),$$

then $(\xi, \xi) = r^2$. Thus ζ must be one of the vectors $\pm \xi / r$. In \mathbf{P}^{n+2} , we have

$$[(\zeta, 1)] = [(\pm \xi / r, 1)] = [(\xi, \pm r)].$$

We can interpret the last coordinate as a signed radius of the sphere with center p and unsigned radius $r > 0$. In order to interpret this geometrically, we adopt the convention that a positive signed radius corresponds to the orientation of the sphere determined by the inward field of unit normals, and a negative signed radius corresponds to the orientation given by the outward field of unit normals. Hence, the two orientations of the sphere in \mathbf{R}^n with center p and unsigned radius $r > 0$ are represented by the two projective points,

$$\left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r \right) \right] \quad (12)$$

in Q^{n+1} . Next if $\zeta_1 + \zeta_2 = 0$, then $[\zeta]$ represents a hyperplane in \mathbf{R}^n , as in equation (8). For $\zeta = (h, -h, N)$, with $|N| = 1$, we have $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \quad (13)$$

These represent the two orientations of the plane with equation $u \cdot N = h$. We make the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N , while the orientation given by $-N$ corresponds to the point $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$.

Thus far we have determined a bijective correspondence between the set of points x in Q^{n+1} with $x_{n+3} \neq 0$ and the set of all oriented spheres and planes in \mathbf{R}^n . Suppose now that $x_{n+3} = 0$, i.e., consider a point $[(z, 0)]$, for $z \in \mathbf{R}_1^{n+2}$. Then $\langle x, x \rangle = (z, z) = 0$, and $[z] \in \mathbf{P}^{n+1}$ is simply a point of the Möbius sphere Σ . Thus we have the following bijective correspondence between objects in Euclidean space and points on the Lie quadric:

Euclidean	Lie	
points: $u \in \mathbf{R}^n$	$\left[\left(\frac{1+u \cdot u}{2}, \frac{1-u \cdot u}{2}, u, 0\right)\right]$	
∞	$[(1, -1, 0, 0)]$	(14)
spheres: center p , signed radius r	$\left[\left(\frac{1+p \cdot p - r^2}{2}, \frac{1-p \cdot p + r^2}{2}, p, r\right)\right]$	
planes: $u \cdot N = h$, unit normal N	$[(h, -h, N, 1)]$	

In Lie sphere geometry, points are considered to be spheres of radius zero, or point spheres. From now on, we will use the term *Lie sphere* or simply “sphere” to denote an oriented sphere, an oriented plane or a point sphere in $\mathbf{R}^n \cup \{\infty\}$. We will refer to the coordinates on the right side of equation (14) as the *Lie coordinates* of the corresponding point, sphere or plane. In the case of \mathbf{R}^2 and \mathbf{R}^3 , respectively, these coordinates were classically called *pentaspherical* and *hexaspherical* coordinates (see [2]).

It is useful to have formulas to convert Lie coordinates back into Cartesian equations for the corresponding Euclidean object. Suppose first that $[x]$ is a point on the Lie quadric with $x_1 + x_2 \neq 0$. Then $x = \rho y$, for some $\rho \neq 0$, where y is one of the standard forms on the right side of the table (14) above. From the table, we see that $y_1 + y_2 = 1$, for all proper points and all spheres. Hence if we divide x by $x_1 + x_2$, the new vector will be in standard form, and we can read off the corresponding Euclidean object from the table. In

particular, if $x_{n+3} = 0$, then $[x]$ represents the point $u = (u_3, \dots, u_{n+2})$ where

$$u_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n + 2. \quad (15)$$

If $x_{n+3} \neq 0$, then $[x]$ represents the sphere with center $p = (p_3, \dots, p_{n+2})$ and signed radius r given by

$$p_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n + 2; \quad r = x_{n+3}/(x_1 + x_2). \quad (16)$$

Finally, suppose that $x_1 + x_2 = 0$. If $x_{n+3} = 0$, then the equation $\langle x, x \rangle = 0$ forces x_i to be zero for $3 \leq i \leq n + 2$. Thus $[x] = [(1, -1, 0, \dots, 0)]$, the improper point. If $x_{n+3} \neq 0$, we divide x by x_{n+3} to make the last coordinate 1. Then if we set $N = (N_3, \dots, N_{n+2})$ and h according to

$$N_i = x_i/x_{n+3}, \quad 3 \leq i \leq n + 2; \quad h = x_1/x_{n+3}, \quad (17)$$

the conditions $\langle x, x \rangle = 0$ and $x_1 + x_2 = 0$ force N to have unit length. Thus $[x]$ corresponds to the hyperplane $u \cdot N = h$, with unit normal N and h as in equation (17).

1.4 Oriented hyperspheres in S^n

In some ways it is simpler to use the sphere S^n rather than \mathbf{R}^n as the base space for the study of Möbius or Lie sphere geometry. This avoids the use of stereographic projection and the need to refer to an improper point or to distinguish between spheres and planes. Furthermore, the correspondence in the table in equation (14) can be reduced to a single formula (21) below.

As in Section 1.2, we consider S^n to be the unit sphere in \mathbf{R}^{n+1} , and then embed \mathbf{R}^{n+1} into \mathbf{P}^{n+1} by the canonical embedding ϕ . Then $\phi(S^n)$ is the Möbius sphere Σ , given by the equation $(z, z) = 0$ in homogeneous coordinates. First we find the Möbius equation for the unoriented hypersphere in S^n with center $p \in S^n$ and spherical radius ρ , for $0 < \rho < \pi$. This hypersphere is the intersection of S^n with the hyperplane in \mathbf{R}^{n+1} given by the equation

$$p \cdot y = \cos \rho, \quad 0 < \rho < \pi. \quad (18)$$

Let $[z] = \phi(y) = [(1, y)]$. Then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)}.$$

Thus equation (18) can be rewritten as

$$(z, (\cos \rho, p)) = 0. \quad (19)$$

Therefore, a point $y \in S^n$ is on the hyperplane determined by equation (18) if and only if $\phi(y)$ lies on the polar hyperplane in \mathbf{P}^{n+1} of the point

$$[\xi] = [(\cos \rho, p)]. \quad (20)$$

To obtain the two oriented spheres determined by equation (18) note that

$$(\xi, \xi) = -\cos^2 \rho + 1 = \sin^2 \rho.$$

Noting that $\sin \rho \neq 0$, we let $\zeta = \pm \xi / \sin \rho$. Then the point $[(\zeta, 1)]$ is on the quadric Q^{n+1} , and

$$[(\zeta, 1)] = [(\xi, \pm \sin \rho)] = [(\cos \rho, p, \pm \sin \rho)].$$

We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0$, $-\pi < \rho < \pi$, and center p corresponds to a point in Q^{n+1} as follows:

$$S \longleftrightarrow [(\cos \rho, p, \sin \rho)]. \quad (21)$$

The formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$. This one formula (21) plays the role of all the formulas given in equation (14) in the preceding section for the Euclidean case.

As in the Euclidean case, the orientation of a sphere S in S^n is determined by a choice of unit normal field to S in S^n . Geometrically, we take the positive radius in (21) to correspond to the field of unit normals which are tangent vectors to geodesics from $-p$ to p . Each oriented sphere can be considered in two ways, with center p and signed radius ρ , $-\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

Given a point $[x]$ in the quadric Q^{n+1} , we now determine the corresponding hypersphere in S^n . Multiplying by -1 , if necessary, we may assume that the first homogeneous coordinate x_1 of x satisfies $x_1 \geq 0$. If $x_1 > 0$, then we see from (21) that the center p and signed radius ρ , $-\pi/2 < \rho < \pi/2$, satisfy

$$\tan \rho = x_{n+3}/x_1, \quad p = (x_2, \dots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \quad (22)$$

If $x_1 = 0$, then $x_{n+3} \neq 0$, so we can divide by x_{n+3} to obtain a point of the form $(0, p, 1)$. This corresponds to the oriented hypersphere with center p and signed radius $\pi/2$, which is a great sphere in S^n .

One can construct the space of oriented hyperspheres in hyperbolic space H^n in a similar way (see [9, p. 18]).

1.5 Oriented contact of spheres

In Möbius geometry, the principal geometric quantity is the angle. In Lie sphere geometry, the corresponding central concept is that of oriented contact of spheres. (See [9, pp. 19–23] for more detail and proofs of the results mentioned in this section.)

Two oriented spheres S_1 and S_2 in \mathbf{R}^n are in *oriented contact* if they are tangent to each other, and they have the same orientation at the point of contact.

If p_1 and p_2 are the respective centers of S_1 and S_2 , and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \quad (23)$$

An oriented sphere S with center p and signed radius r is in oriented contact with an oriented hyperplane π with unit normal N and equation $u \cdot N = h$, if π is tangent to S and their orientations agree at the point of contact. Analytically, this is just the equation

$$p \cdot N = r + h. \quad (24)$$

Two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. Two such planes can be thought of as two oriented spheres in oriented contact at the improper point.

A proper point u in \mathbf{R}^n is in oriented contact with a sphere or plane if it lies on the sphere or plane. Finally, the improper point is in oriented contact with each plane, since it lies on each plane.

Suppose that S_1 and S_2 are two Lie spheres which are represented in the standard form given in equation (14) by $[k_1]$ and $[k_2]$. One can check directly that in all cases, the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \quad (25)$$

It follows from the linear algebra of indefinite scalar product spaces that the Lie quadric contains projective lines but no linear subspaces of \mathbf{P}^{n+2} of higher dimension (see [9, p. 21]). The set of oriented spheres in \mathbf{R}^n corresponding to the points on a line on Q^{n+1} forms a so-called *parabolic pencil* of spheres.

The key result in establishing the relationship between the points on a line in Q^{n+1} and the corresponding parabolic pencil of spheres in \mathbf{R}^n is the following (see [9, p. 22]).

Theorem 1.1. (a) *The line in \mathbf{P}^{n+2} determined by two points $[k_1]$ and $[k_2]$ of Q^{n+1} lies on Q^{n+1} if and only if the spheres corresponding to $[k_1]$ and $[k_2]$ are in oriented contact, i.e., $\langle k_1, k_2 \rangle = 0$.*
(b) *If the line $[k_1, k_2]$ lies on Q^{n+1} , then the parabolic pencil of spheres in \mathbf{R}^n corresponding to points on $[k_1, k_2]$ is precisely the set of all spheres in oriented contact with both $[k_1]$ and $[k_2]$.*

Given any timelike point $[z]$ in \mathbf{P}^{n+2} , the scalar product $\langle \cdot, \cdot \rangle$ has signature $(n+1, 1)$ on z^\perp . Hence, z^\perp intersects Q^{n+1} in a Möbius space. We now show that any line on the quadric intersects such a Möbius space at exactly one point.

Corollary 1.1. *Let $[z]$ be a timelike point in \mathbf{P}^{n+2} and ℓ a line that lies on Q^{n+1} . Then ℓ intersects z^\perp at exactly one point.*

Proof. Any line in projective space intersects a hyperplane in at least one point. We simply must show that ℓ is not contained in z^\perp . This follows from the fact that $\langle \cdot, \cdot \rangle$ has signature $(n+1, 1)$ on z^\perp , and therefore z^\perp cannot contain the 2-dimensional lightlike vector space that projects to ℓ . \square

As a consequence, we obtain the following corollary.

Corollary 1.2. *Every parabolic pencil contains exactly one point sphere. Furthermore, if the point sphere is a proper point, then the pencil contains exactly one plane.*

Proof. The point spheres are precisely the points of intersection of Q^{n+1} with e_{n+3}^\perp . Thus each parabolic pencil contains exactly one point sphere by Corollary 1.1. To prove the second claim, note that the oriented hyperplanes in \mathbf{R}^n correspond to the points in the intersection of Q^{n+1} with $(e_1 - e_2)^\perp$. The line ℓ on the quadric corresponding to the given parabolic pencil intersects

this hyperplane at exactly one point unless ℓ is contained in $(e_1 - e_2)^\perp$. But ℓ is contained in $(e_1 - e_2)^\perp$ if and only if the improper point $[e_1 - e_2]$ is in ℓ^\perp . This implies that the point $[e_1 - e_2]$ is on ℓ . Otherwise, the 2-dimensional linear lightlike subspace of \mathbf{P}^{n+2} spanned by $[e_1 - e_2]$ and ℓ lies on Q^{n+1} , which is impossible. Hence, if the point sphere of the pencil is not the improper point, then the pencil contains exactly one hyperplane. \square

By Corollary 1.2 and Theorem 1.1, we see that if the point sphere in a parabolic pencil is a proper point p in \mathbf{R}^n , then the pencil consists precisely of all spheres in oriented contact with a certain oriented plane π at p . Thus, one can identify the parabolic pencil with the point (p, N) in the unit tangent bundle to \mathbf{R}^n , where N is the unit normal to the oriented plane π . If the point sphere of the pencil is the improper point, then the pencil must consist entirely of planes. Since these planes are all in oriented contact, they all have the same unit normal N . Thus the pencil can be identified with the point (∞, N) in the unit tangent bundle to $\mathbf{R}^n \cup \{\infty\} = S^n$.

It is also useful to have this correspondence between parabolic pencils and elements of the unit tangent bundle $T_1 S^n$ expressed in terms of the spherical metric on S^n . Suppose that ℓ is a line on the quadric. From Corollary 1.1 and equation (21), we see that ℓ intersects both e_1^\perp and e_{n+3}^\perp at exactly one point. So the corresponding parabolic pencil contains exactly one point sphere and one great sphere, represented respectively by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)].$$

The fact that $\langle k_1, k_2 \rangle = 0$ is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p . Hence the parabolic pencil of spheres corresponding to ℓ can be identified with the point (p, ξ) in $T_1 S^n$. The points on the line ℓ can be parametrized as

$$[K_t] = [\cos t k_1 + \sin t k_2] = [(\cos t, \cos t p + \sin t \xi, \sin t)].$$

From equation (21), we see that $[K_t]$ corresponds to the sphere in S^n with center

$$p_t = \cos t p + \sin t \xi, \tag{26}$$

and signed radius t . These are precisely the spheres through p in oriented contact with the great sphere corresponding to $[k_2]$. Their centers lie along the geodesic in S^n with initial point p and initial velocity vector ξ .

1.6 Lie sphere transformations

A *Lie sphere transformation* is a projective transformation of \mathbf{P}^{n+2} which takes Q^{n+1} to itself. In terms of the geometry of \mathbf{R}^n , a Lie sphere transformation maps Lie spheres to Lie spheres. Furthermore, since it is projective, a Lie sphere transformation maps lines on Q^{n+1} to lines on Q^{n+1} . Thus, it preserves oriented contact of spheres in \mathbf{R}^n or S^n .

Pinkall [55] proved the so-called “Fundamental Theorem of Lie sphere geometry,” which states that any line preserving diffeomorphism of Q^{n+1} is the restriction to Q^{n+1} of a projective transformation. Thus, a transformation of the space of oriented spheres which preserves oriented contact must be a Lie sphere transformation. We will not give the proof here, but refer the reader to Pinkall’s paper [55, p. 431] or the book [9, p. 28].

Recall that a linear transformation $A \in GL(n+1)$ induces a projective transformation $P(A)$ on \mathbf{P}^n defined by $P(A)[x] = [Ax]$. The map P is a homomorphism of $GL(n+1)$ onto the group $PGL(n)$ of projective transformations of \mathbf{P}^n . It is well known (see, for example, Samuel [59, p. 6]) that the kernel of P is the group of all nonzero scalar multiples of the identity transformation I .

The group G of Lie sphere transformations is isomorphic to the group $O(n+1, 2)/\{\pm I\}$, where $O(n+1, 2)$ is the group of orthogonal transformations of \mathbf{R}_2^{n+3} . This follows immediately from the following theorem. Here we let $\langle \cdot, \cdot \rangle$ denote the scalar product on \mathbf{R}_k^n (see [9, p. 26] for a proof).

Theorem 1.2. *Let A be a nonsingular linear transformation on the indefinite scalar product space \mathbf{R}_k^n , $1 \leq k \leq n-1$, such that A takes lightlike vectors to lightlike vectors.*

- (a) *Then there is a nonzero constant λ such that $\langle Av, Aw \rangle = \lambda \langle v, w \rangle$ for all v, w in \mathbf{R}_k^n .*
- (b) *Furthermore, if $k \neq n-k$, then $\lambda > 0$.*

Remark 1.1. In the case $k = n-k$, conclusion (b) does not necessarily hold. For example, suppose that $\{v_1, \dots, v_k, w_1, \dots, w_k\}$ is an orthonormal basis for \mathbf{R}_k^{2k} with v_1, \dots, v_k timelike and w_1, \dots, w_k spacelike. Then the linear map T defined by $Tv_i = w_i, Tw_i = v_i$, for $1 \leq i \leq k$, preserves lightlike vectors, but the corresponding λ equals -1 .

From Theorem 1.2 we immediately obtain the following corollary.

Corollary 1.3. (a) *The group G of Lie sphere transformations is isomorphic to $O(n+1, 2)/\{\pm I\}$.*
(b) *The group H of Möbius transformations is isomorphic to $O(n+1, 1)/\{\pm I\}$.*

Proof. (a) Suppose $\alpha = P(A)$ is a Lie sphere transformation. By Theorem 1.2, we have $\langle Av, Aw \rangle = \lambda \langle v, w \rangle$ for all v, w in \mathbf{R}_2^{n+3} , where λ is a positive constant. Set B equal to $A/\sqrt{\lambda}$. Then B is in $O(n+1, 2)$ and $\alpha = P(B)$. Thus, every Lie sphere transformation can be represented by an orthogonal transformation. Conversely, if $B \in O(n+1, 2)$, then $P(B)$ is clearly a Lie sphere transformation. Now let $\Psi : O(n+1, 2) \rightarrow G$ be the restriction of the homomorphism P to $O(n+1, 2)$. Then Ψ is surjective with kernel equal to the intersection of the kernel of P with $O(n+1, 2)$, i.e., kernel $\Psi = \{\pm I\}$.

(b) This follows from Theorem 1.2 in the same manner as (a) with the Lorentz metric being used instead of the Lie metric. \square

Remark 1.2. *On Möbius transformations in Lie sphere geometry.*

A Möbius transformation is a transformation on the space of unoriented spheres, i.e., the space of projective classes of spacelike vectors in \mathbf{R}_1^{n+2} . Hence, each Möbius transformation naturally induces two Lie sphere transformations on the space Q^{n+1} of oriented spheres. Specifically, if A is in $O(n+1, 1)$, then we can extend A to a transformation B in $O(n+1, 2)$ by setting $B = A$ on \mathbf{R}_1^{n+2} and $B(e_{n+3}) = e_{n+3}$. In terms of matrix representation with respect to the standard orthonormal basis, B has the form

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. \quad (27)$$

Note that while A and $-A$ induce the same Möbius transformation, the Lie transformation $P(B)$ is not the same as the Lie transformation $P(C)$ induced by the matrix

$$C = \begin{bmatrix} -A & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} A & 0 \\ 0 & -1 \end{bmatrix},$$

where \simeq denotes equivalence as projective transformations. Hence, the Möbius transformation $P(A) = P(-A)$ induces two Lie transformations, $P(B)$ and $P(C)$. Note that $P(B) = \Gamma P(C)$, where Γ is the *change of orientation transformation* represented in matrix form by

$$\Gamma = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \simeq \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

From equation (14), we see that Γ has the effect of changing the orientation of every oriented sphere or plane.

2 Submanifolds in Lie Sphere Geometry

In this section, we develop the framework necessary to study submanifolds of real space forms within the context of Lie sphere geometry. (See [9, pp. 51–64]) for more detail.) For convenience, we will deal primarily with submanifolds of S^n , and thereby avoid the complications induced by the improper point associated to \mathbf{R}^n . This also makes it easier to treat the important class of isoparametric hypersurfaces in spheres. However, we will also consider submanifolds of \mathbf{R}^n , when it is more convenient to do so.

2.1 Legendre submanifolds

The manifold Λ^{2n-1} of lines on the Lie quadric Q^{n+1} has a contact structure, i.e., a globally defined 1-form ω such that $\omega \wedge (d\omega)^{n-1} \neq 0$ on Λ^{2n-1} . The condition $\omega = 0$ defines a codimension one distribution D on Λ^{2n-1} that has integral submanifolds of dimension $n - 1$, but none of higher dimension (see, for example, [9, p. 57]). Such an integral submanifold of maximal dimension is an immersion $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ satisfying the condition $\lambda^*\omega = 0$. It is called a *Legendre submanifold* [9] or a *Lie geometric hypersurface* [52], [55]. As we shall see, these Legendre submanifolds are natural generalizations of oriented hypersurfaces in S^n or \mathbf{R}^n .

Pinkall [55] (see also [9, pp. 59–60]) showed that a smooth map $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ with $\lambda = [k_1, k_2]$, for smooth maps k_1 and k_2 from M^{n-1} into \mathbf{R}_2^{n+3} , determines a Legendre submanifold if and only if k_1 and k_2 satisfy the following conditions, which we call *Pinkall's Conditions* (1)–(3).

(1) Scalar product conditions: For each $x \in M^{n-1}$, the vectors $k_1(x)$ and $k_2(x)$ are linearly independent and

$$\langle k_1, k_1 \rangle = 0, \quad \langle k_2, k_2 \rangle = 0, \quad \langle k_1, k_2 \rangle = 0.$$

(2) Immersion condition: There is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $dk_1(X)$ and $dk_2(X)$ are both in $Span \{k_1(x), k_2(x)\}$.

(3) Contact condition: $\langle dk_1(X), k_2(x) \rangle = 0$ for all $X \in T_x M^{n-1}$, for all $x \in M^{n-1}$.

These conditions are preserved under a reparametrization $\lambda = [\tilde{k}_1, \tilde{k}_2]$, where $\tilde{k}_1 = \alpha k_1 + \beta k_2$ and $\tilde{k}_2 = \gamma k_1 + \delta k_2$, for smooth functions $\alpha, \beta, \gamma, \delta$ on M^{n-1} with $\alpha\delta - \beta\gamma \neq 0$.

Condition (1) means that the line $[k_1, k_2]$ lies on Q^{n+1} for all $x \in M^{n-1}$, and so λ is a map into Λ^{2n-1} . Condition (2) means that the map λ is an immersion. Condition (3) is equivalent to the contact condition $\lambda^*\omega = 0$.

An oriented hypersurface $f : M^{n-1} \rightarrow S^n$ with field of unit normals $\xi : M^{n-1} \rightarrow S^n$ naturally induces a Legendre submanifold defined by the two functions,

$$k_1(x) = (1, f(x), 0), \quad k_2(x) = (0, \xi(x), 1). \quad (28)$$

For each $x \in M^{n-1}$, $[k_1(x)]$ is the point sphere on the line $\lambda(x)$, and $[k_2(x)]$ is the great sphere on $\lambda(x)$. It is easy to check that the pair $\{k_1, k_2\}$ satisfies the conditions (1)–(3) as follows.

Condition (1) is immediate since both f and ξ are maps into S^n , and $\xi(x)$ is tangent to S^n at $f(x)$ for each x in M^{n-1} . Condition (2) is satisfied since

$$dk_1(X) = (0, df(X), 0),$$

for any vector $X \in T_x M^{n-1}$. Since f is an immersion, $df(X) \neq 0$ for a nonzero vector X , and thus $dk_1(X)$ is not in $\text{Span} \{k_1(x), k_2(x)\}$. Finally, condition (3) is satisfied since for any vector $X \in T_x M^{n-1}$,

$$\langle dk_1(X), k_2(x) \rangle = df(X) \cdot \xi(x) = 0,$$

because ξ is a field of unit normals to f . So in this case, the contact condition (3) just amounts to the fact that ξ is a field of unit normals to f . The map λ is called the *Legendre lift* of the oriented hypersurface f with field of unit normals ξ .

Next, we handle the case of a submanifold $\phi : V \rightarrow S^n$ of codimension greater than one. Let B^{n-1} be the unit normal bundle of the submanifold ϕ . Then B^{n-1} can be considered to be the submanifold of $V \times S^n$ given by

$$B^{n-1} = \{(x, \xi) | \phi(x) \cdot \xi = 0, d\phi(X) \cdot \xi = 0, \text{ for all } X \in T_x V\}.$$

The *Legendre lift* of $\phi(V)$ is the map $\lambda : B^{n-1} \rightarrow \Lambda^{2n-1}$ defined by

$$\lambda(x, \xi) = [k_1(x, \xi), k_2(x, \xi)], \quad (29)$$

where

$$k_1(x, \xi) = (1, \phi(x), 0), \quad k_2(x, \xi) = (0, \xi, 1). \quad (30)$$

Geometrically, $\lambda(x, \xi)$ is the line on the quadric Q^{n+1} corresponding to the parabolic pencil of spheres in S^n in oriented contact at the contact element $(\phi(x), \xi) \in T_1S^n$. In this case, the point sphere map $[k_1]$ has constant rank equal to the dimension of V .

We now return to the case of a general Legendre submanifold $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$. For each $x \in M^{n-1}$, $\lambda(x)$ is a line on the quadric Q^{n+1} . This line contains exactly one point $[k_1(x)]$ corresponding to a point sphere in S^n and one point $[k_2(x)]$ corresponding to a great sphere in S^n . The map $[k_1]$ from M^{n-1} to Q^{n+1} is called the *Möbius projection* or *point sphere map* of λ , and likewise, the map $[k_2]$ is called the *great sphere map*.

The homogeneous coordinates of these points with respect to the standard basis are given by

$$k_1(x) = (1, f(x), 0), \quad k_2(x) = (0, \xi(x), 1), \quad (31)$$

where f and ξ are both smooth maps from M^{n-1} to S^n defined by formula (31). The map f is called the *spherical projection* of λ , and ξ is called the *spherical field of unit normals*. The maps f and ξ depend on the choice of orthonormal basis $\{e_1, \dots, e_{n+3}\}$ for \mathbf{R}_2^{n+3} . For a general Legendre submanifold, the spherical projection f is not necessarily an immersion, nor does it necessarily have constant rank. (See [9, pp. 59–60] for more detail.)

If the range of the point sphere map $[k_1]$ does not contain the improper point $[(1, -1, 0, \dots, 0)]$, then λ also determines a *Euclidean projection*,

$$F : M^{n-1} \rightarrow \mathbf{R}^n,$$

and a *Euclidean field of unit normals*,

$$\eta : M^{n-1} \rightarrow S^{n-1} \subset \mathbf{R}^n.$$

These are defined by the equation $\lambda = [Z_1, Z_2]$, where

$$Z_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Z_2 = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \quad (32)$$

According to (14), $[Z_1(x)]$ corresponds to the unique point sphere in the parabolic pencil determined by $\lambda(x)$, and $[Z_2(x)]$ corresponds to the unique plane in this pencil. As in the spherical case, the smooth maps F and η need not have constant rank. (See [9, pp. 63–64] for more detail.)

2.2 Curvature spheres and Dupin hypersurfaces

To motivate the definition of a curvature sphere we consider the case of an oriented hypersurface $f : M^{n-1} \rightarrow S^n$ with field of unit normals $\xi : M^{n-1} \rightarrow S^n$. The *shape operator* of f at a point $x \in M^{n-1}$ is the symmetric linear transformation $A : T_x M^{n-1} \rightarrow T_x M^{n-1}$ defined by the equation

$$df(AX) = -d\xi(X), \quad X \in T_x M^{n-1}. \quad (33)$$

Often we consider f to be an embedding and suppress the mention of f . Then we identify the tangent vector X with $df(X)$.

The eigenvalues of A are the *principal curvatures*, and the corresponding eigenvectors are the *principal vectors*. We next recall the notion of a focal point of an immersion. For each real number t , define a map

$$f_t : M^{n-1} \rightarrow S^n,$$

by

$$f_t = \cos t f + \sin t \xi. \quad (34)$$

For each $x \in M^{n-1}$, the point $f_t(x)$ lies an oriented distance t along the normal geodesic to $f(M^{n-1})$ at $f(x)$. A point $p = f_t(x)$ is called a *focal point of multiplicity $m > 0$ of f at x* if the nullity of df_t is equal to m at x .

The location of focal points is determined by the principal curvatures. Specifically, if $X \in T_x M^{n-1}$, then by equation (33) we have

$$df_t(X) = \cos t df(X) + \sin t d\xi(X) = df(\cos t X - \sin t AX). \quad (35)$$

Thus, $df_t(X)$ equals zero for $X \neq 0$ if and only if $\cot t$ is a principal curvature of f at x , and X is a corresponding principal vector. Hence, $p = f_t(x)$ is a focal point of f at x of multiplicity m if and only if $\cot t$ is a principal curvature of multiplicity m at x . Note that each principal curvature

$$\kappa = \cot t, \quad 0 < t < \pi,$$

produces two distinct antipodal focal points on the normal geodesic with parameter values t and $t + \pi$.

The oriented hypersphere centered at a focal point p and in oriented contact with $f(M^{n-1})$ at $f(x)$ is called a *curvature sphere* of f at x . The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal

curvatures and curvature spheres is bijective. The multiplicity of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature.

We now consider these ideas as they apply to the Legendre lift of an oriented hypersurface f with field of unit normals ξ . As in equation (28), we have $\lambda = [k_1, k_2]$, where

$$k_1 = (1, f, 0), \quad k_2 = (0, \xi, 1). \quad (36)$$

For each $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be parametrized as

$$[K_t(x)] = [\cos t \, k_1(x) + \sin t \, k_2(x)] = [(\cos t, f_t(x), \sin t)], \quad (37)$$

where f_t is given in equation (34). By equation (21), the point $[K_t(x)]$ in Q^{n+1} corresponds to the oriented sphere in S^n with center $f_t(x)$ and signed radius t . This sphere is in oriented contact with the oriented hypersurface $f(M^{n-1})$ at $f(x)$. Given a tangent vector $X \in T_x M^{n-1}$, we have

$$dK_t(X) = (0, df_t(X), 0). \quad (38)$$

Thus, $dK_t(X) = (0, 0, 0)$ if and only if $df_t(X) = 0$, i.e., $p = f_t(x)$ is a focal point of f at x . Hence, we have shown the following.

Lemma 2.1. *The point $[K_t(x)]$ in Q^{n+1} corresponds to a curvature sphere of the hypersurface f at x if and only if $dK_t(X) = (0, 0, 0)$ for some nonzero vector $X \in T_x M^{n-1}$.*

This characterization of curvature spheres depends on the parametrization of λ given by $\{k_1, k_2\}$ as in equation (36), and it has only been defined in the case where the spherical projection f is an immersion. It is easy to show (see [9, p. 66]) that the following is a generalization of the definition of a curvature sphere that is valid for an arbitrary parametrization of an arbitrary Legendre submanifold.

Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold parametrized by the pair $\{Z_1, Z_2\}$, satisfying Pinkall's Conditions (1)–(3), see page 16. Let $x \in M^{n-1}$ and $r, s \in \mathbf{R}$ with $(r, s) \neq (0, 0)$. The sphere,

$$[K] = [rZ_1(x) + sZ_2(x)],$$

is called a *curvature sphere* of λ at x if there exists a nonzero vector X in $T_x M^{n-1}$ such that

$$r \, dZ_1(X) + s \, dZ_2(X) \in \text{Span} \{Z_1(x), Z_2(x)\}. \quad (39)$$

The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$.

From equation (39), it is clear that the set of principal vectors corresponding to a given curvature sphere $[K]$ at x is a subspace of $T_x M^{n-1}$. This set is called the *principal space* corresponding to the curvature sphere $[K]$. Its dimension is the *multiplicity* of $[K]$.

Remark 2.1. The definition of curvature sphere can be developed in the context of Lie sphere geometry without any reference to submanifolds of S^n (see Cecil–Chern [12] for details). In that case, one begins with a Legendre submanifold $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ and considers a curve $\gamma(t)$ lying in M^{n-1} . The set of points in Q^{n+1} lying on the set of lines $\lambda(\gamma(t))$ forms a ruled surface in Q^{n+1} . One then considers conditions for this ruled surface to be developable. This leads to a system of linear equations whose roots determine the curvature spheres at each point along the curve.

We next want to show that the notion of a curvature sphere is invariant under Lie sphere transformations. Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_2]$. Suppose $\beta = P(B)$ is the Lie sphere transformation induced by an orthogonal transformation B in the group $O(n+1, 2)$. Since B is orthogonal, it is easy to check that the maps, $W_1 = BZ_1$, $W_2 = BZ_2$, satisfy Pinkall’s Conditions (1)–(3) for a Legendre submanifold. We will denote the Legendre submanifold defined by $\{W_1, W_2\}$ by

$$\beta\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}.$$

The Legendre submanifolds λ and $\beta\lambda$ are said to be *Lie equivalent*. In terms of Euclidean geometry, suppose that V and W are two immersed submanifolds of S^n (or \mathbf{R}^n). We say that V and W are *Lie equivalent* if their Legendre lifts are Lie equivalent.

Consider λ and β as above, so that $\lambda = [Z_1, Z_2]$ and $\beta\lambda = [W_1, W_2]$. Note that for a tangent vector $X \in T_x M^{n-1}$ and for real numbers $(r, s) \neq (0, 0)$, we have

$$r dW_1(X) + s dW_2(X) = B(r dZ_1(X) + s dZ_2(X)), \quad (40)$$

since B is linear. Thus, we see that

$$r dW_1(X) + s dW_2(X) \in \text{Span} \{W_1(x), W_2(x)\}$$

if and only if

$$r dZ_1(X) + s dZ_2(X) \in \text{Span} \{Z_1(x), Z_2(x)\}.$$

This immediately implies the following theorem.

Theorem 2.1. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. The point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\beta[K]$ is a curvature sphere of the Legendre submanifold $\beta\lambda$ at x . Furthermore, the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical.*

An important special case is when the Lie sphere transformation is a spherical parallel transformation P_t defined by,

$$\begin{aligned} P_t e_1 &= \cos t e_1 + \sin t e_{n+3}, \\ P_t e_{n+3} &= -\sin t e_1 + \cos t e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \leq i \leq n+2. \end{aligned} \tag{41}$$

The transformation P_t has the effect of adding t to the signed radius of each sphere in S^n while keeping the center fixed (see [9, pp. 48–49]).

Suppose that $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ is a Legendre submanifold parametrized by the point sphere and great sphere maps $\{k_1, k_2\}$, as in equation (36). Then $P_t\lambda = [W_1, W_2]$, where

$$W_1 = P_t k_1 = (\cos t, f, \sin t), \quad W_2 = P_t k_2 = (-\sin t, \xi, \cos t). \tag{42}$$

Note that W_1 and W_2 are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_2 of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t W_1 - \sin t W_2 = (1, \cos t f - \sin t \xi, 0), \\ Z_2 &= \sin t W_1 + \cos t W_2 = (0, \sin t f + \cos t \xi, 1). \end{aligned} \tag{43}$$

From this, we see that $P_t\lambda$ has spherical projection and spherical unit normal field given, respectively, by

$$\begin{aligned} f_{-t} &= \cos t f - \sin t \xi = \cos(-t)f + \sin(-t)\xi, \\ \xi_{-t} &= \sin t f + \cos t \xi = -\sin(-t)f + \cos(-t)\xi. \end{aligned} \tag{44}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ .

Formula (44) shows the close correspondence between these parallel submanifolds and the *parallel hypersurfaces* f_t to f . In the case where the spherical projection f is an immersed hypersurface, a parallel map f_t has a singularity at $x \in M^{n-1}$ if and only if $p = f_t(x)$ is a focal point of M^{n-1} at x . So for each $x \in M^{n-1}$, there are at most $n - 1$ values of $t \in [0, \pi)$ for which f_t fails to be an immersion.

The following theorem, due to Pinkall [55, p. 428] (see also [9, pp. 70–72]), shows that this is also true in the case where the spherical projection f is not an immersion. This theorem is clear if the original spherical projection f is an immersion, but it requires proof if f has singularities. We omit the proof here, however, and refer the reader to [55, p. 428] or [9, pp. 70–72].

Theorem 2.2. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold with spherical projection f and spherical unit normal field ξ . Then for each $x \in M^{n-1}$, the parallel map,*

$$f_t = \cos t f + \sin t \xi,$$

fails to be an immersion at x for at most $n - 1$ values of $t \in [0, \pi)$.

Here $[0, \pi)$ is the appropriate interval, because of the fact mentioned earlier that each principal curvature of an immersion produces two distinct antipodal focal points in the interval $[0, 2\pi)$. We next state some important consequences of this theorem that are obtained by passing to a parallel submanifold, if necessary, and then applying well-known results concerning immersed hypersurfaces in S^n .

Corollary 2.1. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold. Then:*

- (a) *at each point $x \in M^{n-1}$, there are at most $n - 1$ distinct curvature spheres K_1, \dots, K_g ,*
- (b) *the principal vectors corresponding to a curvature sphere K_i form a subspace T_i of the tangent space $T_x M^{n-1}$,*
- (c) *the tangent space $T_x M^{n-1} = T_1 \oplus \dots \oplus T_g$,*
- (d) *if the dimension of a given T_i is constant on an open subset U of M^{n-1} , then the principal distribution T_i is integrable on U ,*
- (e) *if $\dim T_i = m > 1$ on an open subset U of M^{n-1} , then the curvature sphere map K_i is constant along the leaves of the principal foliation T_i .*

Proof. In the case where the spherical projection f of λ is an immersion, the corollary follows from known results concerning hypersurfaces in S^n and the correspondence between the curvature spheres of λ and the principal curvatures of f . Specifically, (a)–(c) follow from elementary linear algebra applied to the (symmetric) shape operator A of the immersion f . As to (d) and (e), Ryan [58, p. 371] showed that the principal curvature functions on an immersed hypersurface are continuous. Nomizu [48] then showed that any continuous principal curvature function κ_i which has constant multiplicity on an open subset U in M^{n-1} is smooth on U , as is its corresponding principal distribution (see also, Singley [63]). If the multiplicity m_i of κ_i equals one on U , then T_i is integrable by the theory of ordinary differential equations. If $m_i > 1$, then the integrability of T_i , and the fact that κ_i is constant along the leaves of T_i are consequences of Codazzi’s equation (Ryan [58], see also Cecil–Ryan [23, p. 24] and Reckziegel [57]).

Note that (a)–(c) are pointwise statements, while (d)–(e) hold on an open set U if they can be shown to hold in a neighborhood of each point of U . Now let x be an arbitrary point of M^{n-1} . If the spherical projection f is not an immersion at x , then by Theorem 2.2, we can find a parallel transformation P_{-t} such that the spherical projection f_t of the Legendre submanifold $P_{-t}\lambda$ is an immersion at x , and hence on a neighborhood of x . By Theorem 2.1, the corollary also holds for λ in this neighborhood of x . Since x is an arbitrary point, the corollary is proved. \square

Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. A connected submanifold S of M^{n-1} is called a *curvature surface* if at each $x \in S$, the tangent space $T_x S$ is equal to some principal space T_i . For example, if $\dim T_i$ is constant on an open subset U of M^{n-1} , then each leaf of the principal foliation T_i is a curvature surface on U . Curvature surfaces are plentiful, since the results of Reckziegel [57] and Singley [63] imply that there is an open dense subset Ω of M^{n-1} on which the multiplicities of the curvature spheres are locally constant. On Ω , each leaf of each principal foliation is a curvature surface.

It is also possible to have a curvature surface S which is not a leaf of a principal foliation, because the multiplicity of the corresponding curvature sphere is not constant on a neighborhood of S , as in the following example of Pinkall [55].

Example 2.1. *A curvature surface that is not a leaf of a principal foliation.* Let T^2 be a torus of revolution in \mathbf{R}^3 , and embed \mathbf{R}^3 into $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$. Let η

be a field of unit normals to T^2 in \mathbf{R}^3 . Let M^3 be a tube of sufficiently small radius $\varepsilon > 0$ around T^2 in \mathbf{R}^4 , so that M^3 is a compact smooth embedded hypersurface in \mathbf{R}^4 . The normal space to T^2 in \mathbf{R}^4 at a point $x \in T^2$ is spanned by $\eta(x)$ and $e_4 = (0, 0, 0, 1)$. The shape operator A_η of T^2 has two distinct principal curvatures at each point of T^2 , while the shape operator A_{e_4} of T^2 is identically zero. Thus the shape operator A_ζ for the normal

$$\zeta = \cos \theta \eta(x) + \sin \theta e_4,$$

at a point $x \in T^2$, is given by

$$A_\zeta = \cos \theta A_{\eta(x)}.$$

From the formulas for the principal curvatures of a tube (see Cecil–Ryan [23, pp. 17–18]), one finds that at all points of M^3 where $x_4 \neq \pm\varepsilon$, there are three distinct principal curvatures of multiplicity one, which are constant along their corresponding lines of curvature (curvature surfaces of dimension one). One of these principal curvatures is $\mu = -1/\varepsilon$ resulting from the tube construction. However, on the two tori, $T^2 \times \{\pm\varepsilon\}$, the principal curvature $\kappa = 0$ has multiplicity two. These two tori are curvature surfaces for this principal curvature κ , since the principal space corresponding to κ is tangent to each torus at every point. These two tori are not leaves of a principal foliation, however, since the leaves of a foliation must all have the same dimension. The Legendre lift λ of this embedding of M^3 in \mathbf{R}^4 has the same properties.

Part (e) of Corollary 2.1 has the following generalization, the proof of which is obtained by invoking the theorem of Ryan [58] mentioned in the proof of Corollary 2.1, with obvious minor modifications.

Corollary 2.2. *Suppose that S is a curvature surface of dimension $m > 1$ in a Legendre submanifold. Then the corresponding curvature sphere is constant along S .*

As we stated at the beginning of the paper, an oriented hypersurface $f : M^{n-1} \rightarrow S^n$ (or \mathbf{R}^n) is said to be *Dupin* if:

- (a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

- (b) the number g of distinct principal curvatures is constant on M .

On an open subset U on which Condition (b) holds, Condition (a) is equivalent to requiring that each curvature surface in each principal foliation be an open subset of a metric sphere in S^n of dimension equal to the multiplicity of the corresponding principal curvature. Condition (a) is also equivalent to the condition that along each curvature surface, the corresponding curvature sphere map is constant. Finally, on U , Condition (a) is equivalent to requiring that for each principal curvature κ , the image of the focal map f_κ is a smooth submanifold of S^n of codimension $m + 1$, where m is the multiplicity of κ . See Cecil–Ryan [23, pp. 18–34] for proofs of these results.

One consequence of the results given above is that like isoparametric hypersurfaces, all proper Dupin hypersurfaces are algebraic. For simplicity, we take the ambient manifold to be \mathbf{R}^n . The theorem was formulated by Cecil, Chi and Jensen [18] as follows.

Theorem 2.3. *Every connected proper Dupin hypersurface $f : M \rightarrow \mathbf{R}^n$ embedded in \mathbf{R}^n is contained in a connected component of an irreducible algebraic subset of \mathbf{R}^n of dimension $n - 1$.*

Pinkall [53] sent the author a letter in 1984 that contained a sketch of a proof of this result. However, a proof was not published until 2008 by Cecil, Chi and Jensen [18], who used methods of real algebraic geometry to give a complete proof based on Pinkall’s sketch. The proof makes use of the various principal foliations whose leaves are open subsets of spheres to construct an analytic algebraic parametrization of a neighborhood of $f(x)$ for each point $x \in M$.

In contrast to the situation for isoparametric hypersurfaces, however, a connected proper Dupin hypersurface in S^n does not necessarily lie in a compact connected proper Dupin hypersurface, because the multiplicities of the principal curvatures are not necessarily constant on a compact Dupin hypersurface that contains the proper Dupin hypersurface (see Remark 2.2 below for an example).

An important class of proper Dupin hypersurfaces consists of the isoparametric hypersurfaces in S^n , and those hypersurfaces in \mathbf{R}^n obtained from

isoparametric hypersurfaces in S^n via stereographic projection. For example, the well-known ring cyclides of Dupin [32] in \mathbf{R}^3 are obtained this way from a standard product torus $S^1(r) \times S^1(s) \subset S^3$, $r^2 + s^2 = 1$.

The torus of revolution in \mathbf{R}^3 in Example 2.1 is a cyclide of Dupin. On the torus, there are $g = 2$ distinct principal curvatures at each point, and each principal curvature is constant along each leaf of its corresponding principal foliation. These leaves are latitude circles for one principal curvature and longitude circles for the other principal curvature.

Remark 2.2. *A Dupin hypersurface that is not proper Dupin.*

The tube $M^3 \subset \mathbf{R}^4$ over the torus in Example 2.1 is an example of a Dupin hypersurface that is not proper Dupin. At points of M^3 except those on the top and bottom tori $T^2 \times \{\pm\varepsilon\}$, there are three distinct principal curvatures that are each constant along their corresponding principal curves (which are circles). However, on $T^2 \times \{\pm\varepsilon\}$, there are only two distinct principal curvatures, $\kappa = 0$ of multiplicity two, and $\mu = -1/\varepsilon$ of multiplicity one. Thus, M^3 is not proper Dupin, since the number of distinct principal curvatures is not constant on M^3 . The hypersurface M^3 is Dupin, however, since along each curvature surface (including $T^2 \times \{\pm\varepsilon\}$), the corresponding principal curvature is constant.

We generalize these definitions to the context of Lie sphere geometry by defining a Legendre submanifold $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ to be a *Dupin submanifold* if:

- (a) along each curvature surface, the corresponding curvature sphere is constant.

The Legendre submanifold λ is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

- (b) the number g of distinct curvature spheres is constant on M .

Of course, the Legendre lift of a Dupin hypersurface in S^n or \mathbf{R}^n is Dupin in the sense defined here, but the definition here is more general, because the spherical projection of a Dupin submanifold need not be an immersion. Corollary 2.2 shows that the only curvature surfaces which must be considered in checking the Dupin property (a) are those of dimension one.

The Legendre lift of the torus of revolution $T^2 \subset \mathbf{R}^3$ in Example 2.1 above is a proper Dupin submanifold. On the other hand, the Legendre lift

of the tube M^3 over T^2 is Dupin, but not proper Dupin, since the number of distinct curvature spheres is not constant on M^3 .

The following theorem shows that both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations, and many important classification results for Dupin submanifolds have been obtained in the setting of Lie sphere geometry.

Theorem 2.4. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation.*

- (a) *If λ is Dupin, then $\beta\lambda$ is Dupin.*
- (b) *If λ is proper Dupin, then $\beta\lambda$ is proper Dupin.*

Proof. By Theorem 2.1, a point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at $x \in M$ if and only if the point $\beta[K]$ is a curvature sphere of $\beta\lambda$ at x , and the principal spaces corresponding $[K]$ and $\beta[K]$ are identical. Since these principal spaces are the same, if S is a curvature surface of λ corresponding to a curvature sphere map $[K]$, then S is also a curvature surface of $\beta\lambda$ corresponding to a curvature sphere map $\beta[K]$, and clearly $[K]$ is constant along S if and only if $\beta[K]$ is constant along S . This proves part (a) of the theorem. Part (b) also follows immediately from Theorem 2.1, since for each $x \in M$, the number g of distinct curvature spheres of λ at x equals the number of distinct curvature spheres of $\beta\lambda$ at x . So if this number g is constant on M for λ , then it is constant on M for $\beta\lambda$. \square

2.3 Local constructions of Dupin hypersurfaces

Pinkall [55] introduced four constructions for obtaining a Dupin hypersurface W in \mathbf{R}^{n+m} from a Dupin hypersurface M in \mathbf{R}^n . We first describe these constructions in the case $m = 1$ as follows.

Begin with a Dupin hypersurface M^{n-1} in \mathbf{R}^n and then consider \mathbf{R}^n as the linear subspace $\mathbf{R}^n \times \{0\}$ in \mathbf{R}^{n+1} . The following constructions yield a Dupin hypersurface W^n in \mathbf{R}^{n+1} .

- (1) Let W^n be the cylinder $M^{n-1} \times \mathbf{R}$ in \mathbf{R}^{n+1} .
- (2) Let W^n be the hypersurface in \mathbf{R}^{n+1} obtained by rotating M^{n-1} around an axis $\mathbf{R}^{n-1} \subset \mathbf{R}^n$.
- (3) Let W^n be a tube of constant radius in \mathbf{R}^{n+1} around M^{n-1} .

- (4) Project M^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. Let W^n be the cone over V^{n-1} in \mathbf{R}^{n+1} .

In general, these constructions introduce a new principal curvature of multiplicity one which is constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of M^{n-1} , and the Dupin property is preserved for these principal curvatures. These constructions can be generalized to produce a new principal curvature of multiplicity m by considering \mathbf{R}^n as a subset of $\mathbf{R}^n \times \mathbf{R}^m$ rather than $\mathbf{R}^n \times \mathbf{R}$. (See [9, pp. 127–141] for a thorough treatment of these constructions in the context of Lie sphere geometry.)

Although Pinkall defined these four constructions, his Theorem 4 [55, p. 438] showed that the cone construction is redundant in the context of Lie sphere geometry, since it is Lie equivalent to a tube (see Remark 2.3 below). For this reason, we will restrict our attention to the three standard constructions: tubes, cylinders and surfaces of revolution.

A Dupin submanifold obtained from a lower-dimensional Dupin submanifold via one of these standard constructions is said to be *reducible*. More generally, a Dupin submanifold which is locally Lie equivalent to such a Dupin submanifold is called *reducible*. Pinkall [55, p. 438] proved the following Lie geometric characterization of reducibility which has been very useful in the study of Dupin hypersurfaces.

Theorem 2.5. *A connected proper Dupin submanifold $\lambda : W^{n-1} \rightarrow \Lambda^{2n-1}$ is reducible if and only if there exists a curvature sphere map $[K]$ of λ that lies in a linear subspace of \mathbf{P}^{n+2} of codimension two.*

Proof. We first note that the following manifolds of spheres are hyperplane sections of the Lie quadric Q^{n+1} :

- a) the hyperplanes in \mathbf{R}^n ,
- b) the spheres with a fixed signed radius r in \mathbf{R}^n ,
- c) the spheres that are orthogonal to a fixed sphere in \mathbf{R}^n .

To see this, we use the Lie coordinates given in equation (14). In Case a), the hyperplanes are characterized by the equation $x_1 + x_2 = 0$, which clearly determines a hyperplane section of Q^{n+1} . In Case b), the spheres with signed

radius r are determined by the linear equation

$$r(x_1 + x_2) = x_{n+3}.$$

In Case c), it can be assumed that the fixed sphere is a hyperplane H through the origin in \mathbf{R}^n . A sphere is orthogonal to H if and only if its center lies in H . This clearly imposes a linear condition on the vector in equation (14) representing the sphere.

The sets a), b), c) are each of the form

$$\{x \in Q^{n+1} \mid \langle x, w \rangle = 0\},$$

with $\langle w, w \rangle = 0, -1, 1$ in Cases a), b), c), respectively.

We can now see that every reducible Dupin hypersurface has a family of curvature spheres that is contained in two hyperplane sections of the Lie quadric as follows.

For the cylinder construction, the tangent hyperplanes of the cylinder are curvature spheres that are orthogonal to a fixed hyperplane in \mathbf{R}^n . Thus, that family of curvature spheres is contained in a n -dimensional linear subspace E of \mathbf{P}^{n+2} such that the signature of $\langle \cdot, \cdot \rangle$ on the polar subspace E^\perp of E is $(0, +)$.

For the surface of revolution construction, the spheres in the new family of curvature spheres all have their centers in the axis of revolution, which is a linear subspace of codimension 2 in \mathbf{R}^n . Thus, that family of curvature spheres is contained in a n -dimensional linear subspace E of \mathbf{P}^{n+2} such that the signature of $\langle \cdot, \cdot \rangle$ on the polar subspace E^\perp of E is $(+, +)$.

For the tube construction, the spheres in the the new family of curvature spheres all have the same radius, and their centers all lie in the hyperplane of \mathbf{R}^n containing the manifold over which the tube is constructed. Thus, that family of curvature spheres is contained in a n -dimensional linear subspace E of \mathbf{P}^{n+2} such that the signature of $\langle \cdot, \cdot \rangle$ on the polar subspace E^\perp of E is $(-, +)$.

Conversely, suppose that $K : W^{n-1} \rightarrow \mathbf{P}^{n+2}$ is a family of curvature spheres that is contained in an n -dimensional linear subspace E of \mathbf{P}^{n+2} . Then $\langle \cdot, \cdot \rangle$ must have signature $(+, +)$, $(0, +)$ or $(-, +)$ on the polar subspace E^\perp , because otherwise $E \cap Q^{n+1}$ would be empty or would consist of a single point.

If the signature of E^\perp is $(+, +)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of all

spheres that have their centers in a fixed $(n - 2)$ -dimensional linear subspace \mathbf{R}^{n-2} of \mathbf{R}^n . Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$, and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ must be a surface of revolution with the axis \mathbf{R}^{n-2} (see [9, pp. 142–143] for more detail on envelopes of families of spheres in this situation), and so λ is reducible.

If the signature of E^\perp is $(0, +)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of hyperplanes orthogonal to a fixed hyperplane in \mathbf{R}^n . Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$, and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ is obtained as a result of the cylinder construction, and so λ is reducible.

If the signature of E^\perp is $(-, +)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of spheres that all have the same radius and whose centers lie in a hyperplane \mathbf{R}^{n-1} of \mathbf{R}^n . Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$, and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ is obtained as a result of the tube construction, and so λ is reducible. \square

Remark 2.3. When Pinkall introduced his constructions, he also listed the following cone construction. Begin with a proper Dupin submanifold λ induced by an embedded proper Dupin hypersurface $M^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. The new Dupin submanifold μ is the Legendre submanifold induced from the cone C^n over M^{n-1} in \mathbf{R}^{n+1} with vertex at the origin. Theorem 2.5 shows that this construction is locally Lie equivalent to the tube construction as follows. As shown in Theorem 2.5, the tube construction is characterized by the fact that one curvature sphere map $[K]$ lies in a n -dimensional linear subspace E of \mathbf{P}^{n+2} , whose orthogonal complement has signature $(-, +)$. For the cone construction, the new family $[K]$ of curvature spheres consists of hyperplanes through the origin (the point $[e_1 + e_2]$ in Lie sphere geometry) that are tangent to the cone along the rulings. Since the hyperplanes all pass through the improper point $[e_1 - e_2]$ as well as the origin $[e_1 + e_2]$, they correspond to points in the linear subspace E , whose orthogonal complement is as follows:

$$E^\perp = \text{Span} \{e_1 + e_2, e_1 - e_2\}.$$

Since E^\perp is spanned by e_1 and e_2 , it has signature $(-, +)$. Thus, the cone construction is Lie equivalent to the tube construction.

Using these constructions, Pinkall was able to produce a proper Dupin hypersurface in Euclidean space with an arbitrary number of distinct principal curvatures, each with any given multiplicity (see Theorem 2.6 below). In general, the proper Dupin hypersurfaces constructed using Pinkall's constructions cannot be extended to compact Dupin hypersurfaces without losing the property that the number of distinct principal curvatures is constant (see [9, pp. 127–141] for more detail). For now, we give a proof of Pinkall's theorem without attempting to compactify the hypersurfaces constructed.

Theorem 2.6. *Given positive integers m_1, \dots, m_g with*

$$m_1 + \dots + m_g = n - 1,$$

there exists a proper Dupin hypersurface in \mathbf{R}^n with g distinct principal curvatures having respective multiplicities m_1, \dots, m_g .

Proof. The proof is by an inductive construction, which will be clear once the first few examples are done. To begin, note that a usual torus of revolution T^2 in \mathbf{R}^3 is a proper Dupin hypersurface with two principal curvatures of multiplicity one. To construct a proper Dupin hypersurface W^3 in \mathbf{R}^4 with three principal curvatures, each of multiplicity one, begin with an open subset U of a torus of revolution in \mathbf{R}^3 on which neither principal curvature vanishes. Take W^3 to be the cylinder $U \times \mathbf{R}$ in $\mathbf{R}^3 \times \mathbf{R} = \mathbf{R}^4$. Then W^3 has three distinct principal curvatures at each point, one of which is zero. These are clearly constant along their corresponding 1-dimensional curvature surfaces.

To get a proper Dupin hypersurface in \mathbf{R}^5 with three principal curvatures having respective multiplicities $m_1 = m_2 = 1$, $m_3 = 2$, one simply takes

$$U \times \mathbf{R}^2 \subset \mathbf{R}^3 \times \mathbf{R}^2 = \mathbf{R}^5,$$

where U is set defined above. To obtain a proper Dupin hypersurface Z^4 in \mathbf{R}^5 with four principal curvatures of multiplicity one, first invert the hypersurface W^3 above in a 3-sphere in \mathbf{R}^4 , chosen so that the image of W^3 contains an open subset V^3 on which no principal curvature vanishes. The hypersurface V^3 is proper Dupin, since the proper Dupin property is preserved by Möbius transformations. Now take Z^4 to be the cylinder $V^3 \times \mathbf{R}$ in $\mathbf{R}^4 \times \mathbf{R} = \mathbf{R}^5$. \square

The proof of this theorem gives an indication of the type of problems that occur when attempting to extend these constructions to produce a compact, proper Dupin hypersurface. In particular, for the cylinder construction, the

new principal curvature on the constructed hypersurface W^n is identically zero. Thus, in order for W^n to be proper Dupin, either zero is not a principal curvature at any point of the original hypersurface M^{n-1} , or else zero is a principal curvature of constant multiplicity on M^{n-1} . Otherwise, the principal curvature zero will not have constant multiplicity on W^n , which implies that W^n is not proper Dupin.

2.4 Lie curvatures of Dupin hypersurfaces

In this section, we introduce certain natural Lie invariants of Legendre submanifolds which have been useful in the study of Dupin and isoparametric hypersurfaces.

Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. As before, we can write $\lambda = [Y_1, Y_2]$ with

$$Y_1 = (1, f, 0), \quad Y_2 = (0, \xi, 1), \quad (45)$$

where f and ξ are the spherical projection and spherical field of unit normals, respectively. At each point $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be written in the form,

$$\mu Y_1(x) + Y_2(x), \quad (46)$$

i.e., take μ as an inhomogeneous coordinate along the projective line $\lambda(x)$. Note that k_1 corresponds to $\mu = \infty$.

The next two theorems give the relationship between the coordinates of the curvature spheres of λ and the principal curvatures of f , in the case where f has constant rank. In the first theorem, we assume that the spherical projection f is an immersion on M^{n-1} . By Theorem 2.2, we know that this can always be achieved locally by passing to a parallel submanifold.

Theorem 2.7. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold whose spherical projection $f : M^{n-1} \rightarrow S^n$ is an immersion. Let Y_1 and Y_2 be the point sphere and great sphere maps of λ as in equation (45). Then the curvature spheres of λ at a point $x \in M^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_2], \quad 1 \leq i \leq g,$$

where $\kappa_1, \dots, \kappa_g$ are the distinct principal curvatures at x of the oriented hypersurface f with field of unit normals ξ . The multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i .

Proof. Let X be a nonzero vector in $T_x M^{n-1}$. Then for any real number μ ,

$$d(\mu Y_1 + Y_2)(X) = (0, \mu df(X) + d\xi(X), 0).$$

This vector is in $\text{Span} \{Y_1(x), Y_2(x)\}$ if and only if

$$\mu df(X) + d\xi(X) = 0,$$

i.e., μ is a principal curvature of f with corresponding principal vector X . \square

A second noteworthy case is when the point sphere map Y_1 is a curvature sphere of constant multiplicity m on M^{n-1} . By Corollary 2.1, the corresponding principal distribution is a foliation, and the curvature sphere map $[Y_1]$ is constant along the leaves of this foliation. Thus the map $[Y_1]$ factors through an immersion $[W_1]$ from the space of leaves V of this foliation into Q^{n+1} . We can write

$$W_1 = (1, \phi, 0),$$

where $\phi : V \rightarrow S^n$ is an immersed submanifold of codimension $m + 1$. The manifold M^{n-1} is locally diffeomorphic to an open subset of the unit normal bundle B^{n-1} of the submanifold ϕ , and λ is essentially the Legendre lift of the submanifold $\phi(V)$, as defined on page 17. The following theorem relates the curvature spheres of λ to the principal curvatures of ϕ . Recall that the point sphere and great sphere maps for λ are given as in equation (30) by

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_2(x, \xi) = (0, \xi, 1), \quad (47)$$

for $(x, \xi) \in B^{n-1}$. The proof of Theorem 2.8 is similar to the proof of Theorem 2.7 above, but it requires putting local coordinates on the unit normal bundle B^{n-1} , and we omit it here (see [9, p. 74]).

Theorem 2.8. *Let $\lambda : B^{n-1} \rightarrow \Lambda^{2n-1}$ be the Legendre lift of the immersed submanifold $\phi(V)$ in S^n of codimension $m + 1$. Let Y_1 and Y_2 be the point sphere and great sphere maps of λ as in equation (47). Then the curvature spheres of λ at a point $(x, \xi) \in B^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_2], \quad 1 \leq i \leq g,$$

where $\kappa_1, \dots, \kappa_{g-1}$ are the distinct principal curvatures of the shape operator A_ξ , and $\kappa_g = \infty$. For $1 \leq i \leq g - 1$, the multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i , while the multiplicity of $[K_g]$ is m .

Given these two theorems, we define a *principal curvature* of a Legendre submanifold $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ at a point $x \in M^{n-1}$ to be a value κ in the set $\mathbf{R} \cup \{\infty\}$ such that $[\kappa Y_1(x) + Y_2(x)]$ is a curvature sphere of λ at x , where Y_1 and Y_2 are as in equation (45).

These principal curvatures of a Legendre submanifold are not Lie invariant, and they depend on the special parametrization for λ given in equation (45). However, Miyaoka [39] pointed out that the cross-ratios of the principal curvatures are Lie invariant.

In order to formulate Miyaoka's theorem, we need to introduce some notation. Suppose that β is a Lie sphere transformation. The Legendre submanifold $\beta\lambda$ has point sphere and great sphere maps given, respectively, by

$$Z_1 = (1, h, 0), \quad Z_2 = (0, \zeta, 1),$$

where h and ζ are the spherical projection and spherical field of unit normals for $\beta\lambda$. Suppose that

$$[K_i] = [\kappa_i Y_1 + Y_2], \quad 1 \leq i \leq g,$$

are the distinct curvature spheres of λ at a point $x \in M^{n-1}$. By Theorem 2.1, the points $\beta[K_i], 1 \leq i \leq g$, are the distinct curvature spheres of $\beta\lambda$ at x . We can write

$$\beta[K_i] = [\gamma_i Z_1 + Z_2], \quad 1 \leq i \leq g.$$

These γ_i are the principal curvatures of $\beta\lambda$ at x .

For four distinct numbers a, b, c, d in $\mathbf{R} \cup \{\infty\}$, we adopt the notation

$$[a, b; c, d] = \frac{(a-b)(d-c)}{(a-c)(d-b)} \quad (48)$$

for the cross-ratio of a, b, c, d . We use the usual conventions involving operations with ∞ . For example, if $d = \infty$, then the expression $(d-c)/(d-b)$ evaluates to one, and the cross-ratio $[a, b; c, d]$ equals $(a-b)/(a-c)$.

Miyaoka's theorem can now be stated as follows.

Theorem 2.9. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. Suppose that $\kappa_1, \dots, \kappa_g, g \geq 4$, are the distinct principal curvatures of λ at a point $x \in M^{n-1}$, and $\gamma_1, \dots, \gamma_g$ are the corresponding principal curvatures of $\beta\lambda$ at x . Then for any choice of four numbers h, i, j, k from the set $\{1, \dots, g\}$, we have*

$$[\kappa_h, \kappa_i; \kappa_j, \kappa_k] = [\gamma_h, \gamma_i; \gamma_j, \gamma_k]. \quad (49)$$

Proof. The left side of equation (49) is the cross-ratio, in the sense of projective geometry, of the four points $[K_h], [K_i], [K_j], [K_k]$ on the projective line $\lambda(x)$. The right side of equation (49) is the cross-ratio of the images of these four points under β . The theorem now follows from the fact that the projective transformation β preserves the cross-ratio of four points on a line. \square

The cross-ratios of the principal curvatures of λ are called the *Lie curvatures* of λ . A set of related invariants for the Möbius group is obtained as follows. First, recall that a Möbius transformation is a Lie sphere transformation that takes point spheres to point spheres. Hence the transformation β in Theorem 2.9 is a Möbius transformation if and only if $\beta[Y_1] = [Z_1]$. This leads to the following corollary of Theorem 2.9.

Corollary 2.3. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold and β a Möbius transformation. Then for any three distinct principal curvatures $\kappa_h, \kappa_i, \kappa_j$ of λ at a point $x \in M^{n-1}$, none of which equals ∞ , we have*

$$\Phi(\kappa_h, \kappa_i, \kappa_j) = (\kappa_h - \kappa_i)/(\kappa_h - \kappa_j) = (\gamma_h - \gamma_i)/(\gamma_h - \gamma_j), \quad (50)$$

where γ_h, γ_i and γ_j are the corresponding principal curvatures of $\beta\lambda$ at the point x .

Proof. First, note that we are using equation (50) to define the quantity Φ . Now since β is a Möbius transformation, the point $[Y_1]$, corresponding to $\mu = \infty$, is taken by β to the point Z_1 with coordinate $\gamma = \infty$. Since β preserves cross-ratios, we have

$$[\kappa_h, \kappa_i; \kappa_j, \infty] = [\gamma_h, \gamma_i; \gamma_j, \infty]. \quad (51)$$

The corollary now follows since the cross-ratio on the left in (51) equals the left side of equation (50), and the cross-ratio on the right in (51) equals the right side of equation (50). \square

A ratio Φ of the form (50) is called a *Möbius curvature* of λ . Lie and Möbius curvatures have been useful in characterizing Legendre submanifolds that are Lie equivalent to Legendre lifts of isoparametric hypersurfaces in spheres, as we will see in the next section.

3 Classifications of Dupin Hypersurfaces

Many of the important classification results for Dupin hypersurfaces involve conditions under which the Dupin hypersurface is Lie equivalent to the Legendre lift of an isoparametric hypersurface in a sphere. These results will be discussed in this section. We begin with a brief summary of the classification results for isoparametric hypersurfaces in spheres.

3.1 Isoparametric hypersurfaces in spheres

An oriented hypersurface M in a real space form \mathbf{R}^n , S^n or H^n is called an isoparametric hypersurface if it has constant principal curvatures (see, for example, [23, pp. 1–3] for other characterizations).

An isoparametric hypersurface M in \mathbf{R}^n can have at most two distinct principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbf{R}^{n-k-1}$. This was first proven for $n = 3$ by Somigliana [64] in 1919 (see also Levi-Civita [36] (1937) for $n = 3$ and B. Segre [60] (1938) for arbitrary n).

Shortly after the publication of the papers of Levi-Civita and Segre, Cartan [3]–[6] undertook the study of isoparametric hypersurfaces in arbitrary real space-forms. Cartan showed that an isoparametric hypersurface in \mathbf{R}^n or H^n can have at most two distinct principal curvatures, and he gave a local classification of isoparametric hypersurfaces in both ambient spaces (see [23, pp. 85–101]).

In the sphere S^n , however, Cartan showed that there are many more possibilities. He found examples of isoparametric hypersurfaces in S^n with 1, 2, 3 or 4 distinct principal curvatures, and he classified connected isoparametric hypersurfaces with $g \leq 3$ principal curvatures as follows. If $g = 1$, then the isoparametric hypersurface M is totally umbilic, and it must be an open subset of a great or small sphere. If $g = 2$, then M must be an open subset of a standard product of two spheres,

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1.$$

In the case $g = 3$, Cartan [4] showed that all the principal curvatures must have the same multiplicity $m = 1, 2, 4$ or 8 , and the isoparametric hypersurface must be an open subset of a tube of constant radius over a standard embedding of a projective plane $\mathbf{F}P^2$ into S^{3m+1} (see, for example, Cecil–Ryan [23, pp. 151–155]), where \mathbf{F} is the division algebra \mathbf{R} , \mathbf{C} , \mathbf{H}

(quaternions), \mathbf{O} (Cayley numbers), for $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m .

Cartan's theory was further developed by Nomizu [49]–[50], Takagi and Takahashi [67], Ozeki and Takeuchi [51], and most extensively by Münzner [44]–[45] (see also the English translations [44]–[45]), who showed that the number g of distinct principal curvatures of an isoparametric hypersurface must be 1, 2, 3, 4 or 6. For more detail on the theory of isoparametric hypersurfaces, see Chapter 3 of Cecil–Ryan [23] or Chapter 3 of [22].

In the case of an isoparametric hypersurface with four principal curvatures, Münzner proved that the principal curvatures can have at most two distinct multiplicities m_1, m_2 . Next Ferus, Karcher and Münzner [33] (see also the English translation [33]) used representations of Clifford algebras to construct for any positive integer m_1 an infinite series of isoparametric hypersurfaces with four principal curvatures having respective multiplicities (m_1, m_2) , where m_2 is nondecreasing and unbounded in each series. (These examples are also described in [9, pp. 95–112] and [23, pp. 162–180].)

As later work by several researchers would show (see below), this class of *FKM-type* isoparametric hypersurfaces contains all isoparametric hypersurfaces with four principal curvatures with the exception of two homogeneous examples, having multiplicities $(2, 2)$ and $(4, 5)$. This construction of Ferus, Karcher and Münzner was a generalization of an earlier construction due to Ozeki and Takeuchi [51].

Stolz [65] next proved that the multiplicities (m_1, m_2) of the principal curvatures of an isoparametric hypersurface with four principal curvatures must be the same as those of the hypersurfaces of FKM-type or the two homogeneous exceptions. Cecil, Chi and Jensen [15] then showed that if the multiplicities of an isoparametric hypersurface with four principal curvatures satisfy $m_2 \geq 2m_1 - 1$, then the hypersurface is of FKM-type. (A different proof of this result, using isoparametric triple systems, was given later by Immervoll [35].)

Taken together with known results of Takagi [66] for $m_1 = 1$, and Ozeki and Takeuchi [51] for $m_1 = 2$, this result of Cecil, Chi and Jensen handled all possible pairs of multiplicities except for four cases, the homogeneous pair $(4, 5)$, and the FKM pairs $(3, 4)$, $(6, 9)$ and $(7, 8)$. In a series of recent papers, Chi [24]–[29] completed the classification of isoparametric hypersurfaces with four principal curvatures. Specifically, Chi showed that in the cases $(3, 4)$, $(6, 9)$ and $(7, 8)$, the isoparametric hypersurface must be of FKM-type, and in the case $(4, 5)$, it must be homogeneous.

In the case of an isoparametric hypersurface with six principal curvatures, Münzner showed that all of the principal curvatures must have the same multiplicity m , and Abresch [1] showed that m must equal 1 or 2. By the classification of homogeneous isoparametric hypersurfaces due to Takagi and Takahashi [67], there is only one homogeneous family in each case up to congruence. These homogeneous examples have been shown to be the only isoparametric hypersurfaces in the case $g = 6$ by Dorfmeister and Neher [31] in the case of multiplicity $m = 1$, and by Miyaoka [41]–[42] in the case $m = 2$ (see also the papers of Siffert [61]–[62]). See the surveys of Thorbergsson [69], Cecil [10], and Chi [30] for more information on isoparametric hypersurfaces in spheres.

3.2 Compact proper Dupin hypersurfaces

There is a close relationship between the theory of isoparametric hypersurfaces and the theory of compact proper Dupin hypersurfaces embedded in S^n (or \mathbf{R}^n). Thorbergsson [68] showed that the restriction $g = 1, 2, 3, 4$ or 6 on the number of distinct principal curvatures of an isoparametric hypersurface also holds for a connected, compact proper Dupin hypersurface M embedded in $S^n \subset \mathbf{R}^{n+1}$. He first showed that M must be *taut*, i.e., every nondegenerate distance function $L_p(x) = d(p, x)^2$, where $d(p, x)$ is the distance from p to x in S^n , has the minimum number of critical points required by the Morse inequalities on M . Using tautness, he then showed that M divides S^n into two ball bundles over the first focal submanifolds on either side of M . This topological situation is what is required for Münzner’s proof of the restriction on g . (See also [23, 65–74].)

Münzner’s argument also produces certain restrictions on the cohomology and the homotopy groups of isoparametric hypersurfaces. These restrictions necessarily apply to compact proper Dupin hypersurfaces by Thorbergsson’s result. Grove and Halperin [34] later found more topological similarities between these two classes of hypersurfaces. Furthermore, the results of Stolz [65] and Grove-Halperin [34] on the possible multiplicities of the principal curvatures actually only require the assumption that M is a compact proper Dupin hypersurface, and not that the hypersurface is isoparametric.

The close relationship between these two classes of hypersurfaces led to the widely held conjecture that every compact proper Dupin hypersurface $M \subset S^n$ is Lie equivalent to an isoparametric hypersurface (see [22, p. 184]). We now describe the results that have been obtained concerning this con-

ture. (See [11] for a full account.)

The conjecture is obviously true for $g = 1$, in which case the compact proper Dupin hypersurface $M \subset S^n$ must be a hypersphere in S^n , and so M itself is isoparametric. In 1978, Cecil and Ryan [21] showed that if $g = 2$, then M must be an $(n - 1)$ -dimensional cyclide of Dupin, and it is therefore Möbius equivalent to an isoparametric hypersurface in S^n . Then in 1984, Miyaoka [38] showed that the conjecture is true for $g = 3$, that is, M must be Lie equivalent an isoparametric hypersurface, although M is not necessarily Möbius equivalent to an isoparametric hypersurface. Thus, as g increases, the group needed to obtain equivalence with an isoparametric hypersurface gets progressively larger.

For $g = 4$, the conjecture remained open for several years until finally in 1988, counterexamples to the conjecture were discovered independently by Pinkall and Thorbergsson [56] and by Miyaoka and Ozawa [43]. The two types of counterexamples are different, and the method of Miyaoka and Ozawa also yields counterexamples to the conjecture with $g = 6$ principal curvatures.

For both the counterexamples of Miyaoka-Ozawa and those of Pinkall-Thorbergsson, the hypersurfaces constructed do not have constant Lie curvatures, and so they cannot be Lie equivalent to an isoparametric hypersurface. (See also [9, pp. 112–123] or [23, pp. 308–322] for a description of these counterexamples to the conjecture.)

This led to the following revised version of the conjecture due to Cecil, Chi and Jensen [16], [17] that includes the assumption of constant Lie curvatures.

Conjecture 3.1. *Every compact, connected proper Dupin hypersurface in S^n with $g = 4$ or $g = 6$ principal curvatures and constant Lie curvatures is Lie equivalent to an isoparametric hypersurface.*

Research on this revised conjecture has been an important part of the field, since the publication of the counterexamples to the original conjecture. A useful result in this area is the following local Lie geometric characterization of Legendre submanifolds that are Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n (see [7] or [9, p. 77]).

Recall that a line in \mathbf{P}^{n+2} is called *timelike* if it contains only timelike points. This means that an orthonormal basis for the 2-plane in \mathbf{R}_2^{n+3} determined by the timelike line in \mathbf{P}^{n+2} consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$.

Theorem 3.1. *Let $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $[K_1], \dots, [K_g]$ at each point. Then λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n if and only if there exist g points $[P_1], \dots, [P_g]$ on a timelike line in \mathbf{P}^{n+2} such that*

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Proof. If λ is the Legendre lift of an isoparametric hypersurface in S^n , then all the spheres in each family $[K_i]$ have the same radius ρ_i , where $0 < \rho_i < \pi$. By formula (21), this is equivalent to the condition $\langle K_i, P_i \rangle = 0$, where the

$$P_i = \sin \rho_i e_1 - \cos \rho_i e_{n+3}, \quad 1 \leq i \leq g, \quad (52)$$

are g points on the timelike line $[e_1, e_{n+3}]$. Since a Lie sphere transformation preserves curvature spheres, timelike lines and the polarity relationship, the same is true for any image of λ under a Lie sphere transformation.

Conversely, suppose that there exist g points $[P_1], \dots, [P_g]$ on a timelike line ℓ such that

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Let β be a Lie sphere transformation that maps ℓ to the line $[e_1, e_{n+3}]$. Then the curvature spheres $\beta[K_i]$ of $\beta\lambda$ are respectively orthogonal to the points $[Q_i] = \beta[P_i]$ on the line $[e_1, e_{n+3}]$. This means that for each value of i , the spheres corresponding to $\beta[K_i]$ have constant radius on M^{n-1} . By applying a parallel transformation, if necessary, we can arrange that none of these curvature spheres has radius zero. Then $\beta\lambda$ is the Legendre lift of an isoparametric hypersurface in S^n . \square

Remark 3.1. In the case where λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n , one can say more about the position of the points $[P_1], \dots, [P_g]$ on the timelike line ℓ . Münzner [44]–[45] showed that the radii ρ_i of the curvature spheres of an isoparametric hypersurface must be of the form

$$\rho_i = \rho_1 + (i-1)\frac{\pi}{g}, \quad 1 \leq i \leq g, \quad (53)$$

for some $\rho_1 \in (0, \pi/g)$. Hence, after Lie sphere transformation, the $[P_i]$ must have the form (52) for ρ_i as in equation (53).

Since the principal curvatures are constant on an isoparametric hypersurface, the Lie curvatures are also constant. By Münzner's work, the distinct principal curvatures $\kappa_i, 1 \leq i \leq g$, of an isoparametric hypersurface must have the form

$$\kappa_i = \cot \rho_i, \quad (54)$$

for ρ_i as in equation (53). Thus the Lie curvatures of an isoparametric hypersurface can be determined. We can order the principal curvatures so that

$$\kappa_1 < \cdots < \kappa_g. \quad (55)$$

In the case $g = 4$, this leads to a unique Lie curvature Ψ defined by

$$\Psi = [\kappa_1, \kappa_2; \kappa_3, \kappa_4] = (\kappa_1 - \kappa_2)(\kappa_4 - \kappa_3)/(\kappa_1 - \kappa_3)(\kappa_4 - \kappa_2). \quad (56)$$

The ordering of the principal curvatures implies that Ψ must satisfy $0 < \Psi < 1$. Using equations (54) and (56), one can compute that $\Psi = 1/2$ on any isoparametric hypersurface, i.e., the four curvature spheres form a *harmonic set* in the sense of projective geometry (see, for example, [59, p. 59]).

There is, however, a simpler way to compute Ψ . One applies Theorem 2.8 to the Legendre lift of one of the focal submanifolds of the isoparametric hypersurface. By the work of Münzner, each isoparametric hypersurface M^{n-1} embedded in S^n has two distinct focal submanifolds, each of codimension greater than one. The hypersurface M^{n-1} is a tube of constant radius over each of these focal submanifolds. Therefore, the Legendre lift of M^{n-1} is obtained from the Legendre lift of either focal submanifold by parallel transformation. Thus, the Legendre lift of M^{n-1} has the same Lie curvature as the Legendre lift of either focal submanifold. Let $\phi : V \rightarrow S^n$ be one of the focal submanifolds. By the same calculation that yields equation (53), Münzner showed that if ξ is any unit normal to $\phi(V)$ at any point, then the shape operator A_ξ has three distinct principal curvatures,

$$\kappa_1 = -1, \quad \kappa_2 = 0, \quad \kappa_3 = 1.$$

By Theorem 2.8, the Legendre lift of ϕ has a fourth principal curvature $\kappa_4 = \infty$. Thus, the Lie curvature of this Legendre submanifold is

$$\Psi = (-1 - 0)(\infty - 1)/(-1 - 1)(\infty - 0) = 1/2.$$

In the case $g = 4$, one can ask what is the strength of the assumption $\Psi = 1/2$ on M^{n-1} . Since Ψ is only one function of the principal curvatures,

one would not expect this assumption to classify Legendre submanifolds up to Lie equivalence. However, if one makes additional assumptions, e.g., the Dupin condition, then results can be obtained.

Miyaoka [39] proved that the assumption that Ψ is constant on a compact connected proper Dupin hypersurface M^{n-1} in S^n with four principal curvatures, together with an additional assumption regarding intersections of leaves of the various principal foliations, implies that M^{n-1} is Lie equivalent to an isoparametric hypersurface. This proved that Conjecture 3.1 is true under Miyaoka's additional assumptions. However, no one has proved that Miyaoka's additional assumptions always hold. In the same paper, Miyaoka also proved that if the Lie curvature of a compact proper Dupin hypersurface with $g = 4$ is constant, then it has the value $1/2$.

As mentioned above, Thorbergsson [68] showed that for a compact proper Dupin hypersurface in S^n with four principal curvatures, the multiplicities of the principal curvatures must satisfy $m_1 = m_3, m_2 = m_4$, when the principal curvatures are ordered as in equation (55), the same restriction as for an isoparametric hypersurface (see, for example, [23, p.108]).

Cecil, Chi and Jensen [16] then used a different approach than Miyaoka [39] to try to prove Conjecture 3.1. First, they proved that the Legendre lift of a connected, compact proper Dupin hypersurface with $g > 2$ principal curvatures is irreducible (in the sense of Pinkall [55], see Section 2.3). Then they worked locally using the method of moving frames and assuming irreducibility to prove the following theorem.

Theorem 3.2. *Let M be an irreducible connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_3 \geq 1$, $m_2 = m_4 = 1$, and constant Lie curvature $\Psi = 1/2$. Then M is Lie equivalent to an open subset of an isoparametric hypersurface.*

Theorem 3.2 implies the following result in the compact case due to Cecil, Chi and Jensen [16]:

Theorem 3.3. *Let M be a compact, connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_3 \geq 1$, $m_2 = m_4 = 1$, and constant Lie curvature. Then M is Lie equivalent to an isoparametric hypersurface.*

Theorem 3.2 implies Theorem 3.3 for the following reasons. As noted above, Cecil, Chi and Jensen [16] proved that the Legendre lift of a connected, compact proper Dupin hypersurface with $g > 2$ principal curvatures

is irreducible. Next Thorbergsson's [68] result proves that the multiplicities of a compact, connected proper Dupin hypersurface with four principal curvatures satisfy the conditions $m_1 = m_3$ and $m_2 = m_4$, when the principal curvatures are appropriately ordered. Finally, Miyaoka [39] proved that if the Lie curvature of a compact proper Dupin hypersurface with $g = 4$ is constant, then it has the value $1/2$. Thus, the hypotheses of Theorem 3.3 imply that the hypotheses of Theorem 3.2 are satisfied.

This means that the full Conjecture 3.1 for $g = 4$ would be proven, if the assumption that the value of $m_2 = m_4$ is equal to one could be eliminated from Theorem 3.2, and thus from Theorem 3.3.

The proof of Theorem 3.2 involves some complicated calculations, which would become even more elaborate if the value of $m_2 = m_4$ were allowed to be greater than one. Even so, this approach to proving Conjecture 3.1 could possibly be successful with some additional insight regarding the structure of the calculations involved. The criteria for reducibility (Theorem 2.5) and the criteria for Lie equivalence to an isoparametric hypersurface (Theorem 3.1) are both important in the proof of Theorem 3.2.

In the case $g = 6$, we do not know of any results beyond those of Miyaoka [40], who showed that if the Dupin hypersurface satisfies some additional assumptions on the intersections of the leaves of the various principal foliations, then Conjecture 3.1 is true for $g = 6$. However, no one has proved that Miyaoka's additional assumptions always hold.

A local approach assuming irreducibility, similar to that used by Cecil, Chi and Jensen for the $g = 4$ case, might possibly work for the case $g = 6$, but the calculations involved would be very complicated, unless some new algebraic insight is found to simplify the situation.

3.3 Examples with constant Lie curvature that are not Lie equivalent to an isoparametric hypersurface

The following example [7] (see also [9, pp. 80–82]) is a noncompact proper Dupin hypersurface with $g = 4$ distinct principal curvatures of multiplicity one, and constant Lie curvature $\Psi = 1/2$, which is not Lie equivalent to an open subset of an isoparametric hypersurface with four principal curvatures in S^n .

This example is reducible in the sense of Pinkall (see Section 2.3), and it cannot be made compact without destroying the property that the number

g of distinct curvatures spheres equals four at each point. This example shows that some additional hypotheses (either compactness or irreducibility) besides constant Lie curvature $\Psi = 1/2$ are needed to conclude that a proper Dupin hypersurface with $g = 4$ principal curvatures is Lie equivalent to an isoparametric hypersurface.

Example 3.1. Let $\phi : V \rightarrow S^{n-m}$ be an embedded Dupin hypersurface in S^{n-m} with field of unit normals ξ in S^{n-m} , such that ϕ has three distinct principal curvatures,

$$\mu_1 < \mu_2 < \mu_3,$$

at each point of V . Embed S^{n-m} as a totally geodesic submanifold of S^n , and let B^{n-1} be the unit normal bundle of the submanifold $\phi(V)$ in S^n . Let $\lambda : B^{n-1} \rightarrow \Lambda^{2n-1}$ be the Legendre lift of the submanifold $\phi(V)$ in S^n . Any unit normal η to $\phi(V)$ at a point $x \in V$ can be written in the form

$$\eta = \cos \theta \xi(x) + \sin \theta \zeta,$$

where ζ is a unit normal to S^{n-m} in S^n . Since the shape operator $A_\zeta = 0$, we have

$$A_\eta = \cos \theta A_\xi.$$

Thus the principal curvatures of A_η are

$$\kappa_i = \cos \theta \mu_i, \quad 1 \leq i \leq 3. \quad (57)$$

If $\eta \cdot \xi = \cos \theta \neq 0$, then A_η has three distinct principal curvatures. However, if $\eta \cdot \xi = 0$, then $A_\eta = 0$.

Let U be the open subset of B^{n-1} on which $\cos \theta > 0$, and let α denote the restriction of λ to U . By Theorem 2.8, α has four distinct curvature spheres at each point of U . Since $\phi(V)$ is Dupin in S^{n-m} , it is easy to show that α is Dupin (see the tube construction [9, pp. 127-133] for the details). Furthermore, since $\kappa_4 = \infty$, the Lie curvature Ψ of α at a point (x, η) of U equals the Möbius curvature $\Phi(\kappa_1, \kappa_2, \kappa_3)$. Using equation (57), we compute

$$\Psi = \Phi(\kappa_1, \kappa_2, \kappa_3) = \frac{\kappa_1 - \kappa_2}{\kappa_1 - \kappa_3} = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3} = \Phi(\mu_1, \mu_2, \mu_3). \quad (58)$$

Now consider the special case where $\phi(V)$ is a minimal isoparametric hypersurface in S^{n-m} with three distinct principal curvatures. By Münzner's formula (53), these principal curvatures must have the values,

$$\mu_1 = -\sqrt{3}, \quad \mu_2 = 0, \quad \mu_3 = \sqrt{3}.$$

On the open subset U of B^{n-1} described above, the Lie curvature of α has the constant value $1/2$ by equation (58). To construct an immersed proper Dupin hypersurface with four principal curvatures and constant Lie curvature $\Psi = 1/2$ in S^n , we simply take the open subset $P_{-t}\alpha(U)$ of the tube of radius t around $\phi(V)$ in S^n for an appropriate value of t .

We can see that this example is not Lie equivalent to an open subset of an isoparametric hypersurface in S^n with four distinct principal curvatures as follows. Note that the example that we have constructed is reducible, because it is obtained by the tube construction from the hypersurface $\phi(V)$ in $S^{n-m} \subset S^n$. On the other hand, as we will show in the next paragraph, any open subset of an isoparametric hypersurface with four principal curvatures in S^n is irreducible, so it cannot be Lie equivalent to our example.

The irreducibility of an open subset of an isoparametric hypersurface with four principal curvatures in S^n follows from the results of Cecil, Chi and Jensen [16], who proved that a compact proper Dupin hypersurface with $g > 2$ principal curvatures is irreducible. Furthermore, they proved that an irreducible proper Dupin hypersurface cannot contain a reducible open subset. This follows from the algebraicity of proper Dupin hypersurfaces, as stated in Theorem 2.3 (see also [9, pp. 145-147]). Thus, a compact isoparametric hypersurface with four principal curvatures in S^n is irreducible, and it cannot contain a reducible open subset.

Note that the point sphere map $[Y_1]$ of the Legendre submanifold $\lambda : B^{n-1} \rightarrow \Lambda^{2n-1}$ defined above is a curvature sphere of multiplicity m which lies in the linear subspace of codimension $m + 1$ in \mathbf{P}^{n+2} orthogonal to the space spanned by e_{n+3} and by those vectors ζ normal to S^{n-m} in S^n . This geometric fact implies that for such a vector ζ , there are only two distinct curvature spheres on each of the lines $\lambda(x, \zeta)$, since $A_\zeta = 0$ (see Theorem 2.8). This change in the number of distinct curvature spheres at points of the form (x, ζ) is precisely why α cannot be extended to a compact proper Dupin submanifold with $g = 4$.

With regard to Theorem 3.1, α comes as close as possible to satisfying the requirements for being Lie equivalent to an isoparametric hypersurface without actually fulfilling them. The principal curvatures $\kappa_2 = 0$ and $\kappa_4 = \infty$ are constant on U . If a third principal curvature were also constant, then the constancy of Ψ would imply that all four principal curvatures were constant, and α would be the Legendre lift of an isoparametric hypersurface.

Using this same method, it is easy to construct reducible noncompact

proper Dupin hypersurfaces in S^n with $g = 4$ and $\Psi = c$, for any constant $0 < c < 1$. If $\phi(V)$ is an isoparametric hypersurface in S^{n-m} with three distinct principal curvatures, then Münzner's formula (53) implies that these principal curvatures must have the values,

$$\mu_1 = \cot\left(\theta + \frac{2\pi}{3}\right), \quad \mu_2 = \cot\left(\theta + \frac{\pi}{3}\right), \quad \mu_3 = \cot \theta, \quad 0 < \theta < \frac{\pi}{3}. \quad (59)$$

Furthermore, any value of θ in $(0, \pi/3)$ can be realized by some hypersurface in a parallel family of isoparametric hypersurfaces in S^{n-m} . A direct calculation using equations (58) and (59) shows that the Lie curvature Ψ of α satisfies

$$\Psi = \Phi(\kappa_1, \kappa_2, \kappa_3) = \frac{\kappa_1 - \kappa_2}{\kappa_1 - \kappa_3} = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3} = \frac{1}{2} + \frac{\sqrt{3}}{2} \tan\left(\theta - \frac{\pi}{6}\right)$$

on U . This can assume any value c in the interval $(0, 1)$ by an appropriate choice of θ in $(0, \pi/3)$. An open subset of a tube over $\phi(V)$ in S^n is a proper Dupin hypersurface with $g = 4$ and $\Psi = \Phi = c$. Note that Φ has different values on different hypersurfaces in the parallel family of isoparametric hypersurfaces in S^{n-m} . Thus these hypersurfaces are not Möbius equivalent to each other by Corollary 2.3. This is consistent with the fact that a parallel transformation is not a Möbius transformation.

3.4 Local classifications of Dupin hypersurfaces

In this section, we mention local classifications of proper Dupin hypersurfaces that have been obtained using the methods of Lie sphere geometry. In the case of $g = 2$ principal curvatures, the classification of Dupin surfaces in S^3 (the cyclides of Dupin) goes back to the nineteenth century. (See, [22, pp. 151–166] or [9, p. 148] for more detail.)

A proper Dupin hypersurface $M \subset S^n$ with two distinct curvature spheres of respective multiplicities p and q is called a *cyclide of Dupin of characteristic (p, q)* (see Pinkall [55]). In the same paper, Pinkall [55] proved the following classification of the higher dimensional cyclides (see also [9, pp. 149–151] for a proof).

Theorem 3.4. (a) *Every connected cyclide of Dupin of characteristic (p, q) is contained in a unique compact, connected cyclide of Dupin of characteristic (p, q) .*

- (b) *Any two cyclides of Dupin of the same characteristic (p, q) are locally Lie equivalent, each being Lie equivalent to an open subset of a standard product of two spheres*

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \quad (60)$$

where $p + q = n - 1$.

From this, one can derive a Möbius geometric classification of the cyclides of Dupin as follows (see [8], [9, pp. 151–159]).

Theorem 3.5. (a) *Every connected cyclide of Dupin $M^{n-1} \subset \mathbf{R}^n$ of characteristic (p, q) is Möbius equivalent to an open subset of a hypersurface of revolution obtained by revolving a q -sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ or a p -sphere $S^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^p \subset \mathbf{R}^{p+1}$.*

(b) *Two hypersurfaces obtained by revolving a q -sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis of revolution $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ are Möbius equivalent if and only if they have the same value of $\rho = |r|/a$, where r is the signed radius of the profile sphere S^q and $a > 0$ is the distance from the center of S^q to the axis of revolution.*

In the case $g = 3$, Pinkall [52], [54] (see also the English translation [52]), obtained a classification up to Lie equivalence of proper Dupin hypersurfaces in \mathbf{R}^4 with three distinct principal curvatures of multiplicity one. Another version of Pinkall's proof in the irreducible case was given later by Cecil and Chern [13] (see also [14], [9, pp. 168–190]). Cecil and Chern developed the method of moving Lie frames which can be applied to the general study of Legendre submanifolds. This approach has been applied successfully by Niebergall [46]–[47], Cecil and Jensen [19]–[20], and by Cecil, Chi and Jensen [16] to obtain local classifications of higher-dimensional Dupin hypersurfaces.

In particular, Cecil and Jensen [19] proved that for a connected irreducible proper Dupin hypersurface with three principal curvatures, all of the multiplicities must be equal, and the hypersurface must be Lie equivalent to an open subset of an isoparametric hypersurface in a sphere with three principal curvatures.

An open problem is the classification of reducible Dupin hypersurfaces of arbitrary dimension with $g = 3$ principal curvatures up to Lie equivalence. Pinkall [52], [54], found such a classification in the case $M^3 \subset \mathbf{R}^4$, and it may be possible to obtain results in higher dimensions using his approach.

As noted in Section 3.2, local classification in the case $g = 4$ has been studied extensively in the papers of Niebergall [46]–[47], Cecil and Jensen [19]–[20], and by Cecil, Chi and Jensen [16], culminating in Theorem 3.2 of Section 3.2. The method of moving frames is central to the approach of those papers.

As to local classifications in the case $g = 6$, an approach similar to that of Cecil, Chi and Jensen [16] is plausible, but the calculations involved would be very complicated, unless some new algebraic insight is found to simplify the situation.

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