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Riemann Surfaces: Distribution of Weierstrass Points on Nodal Curves of Genus 2

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I. Weierstrass Points on Riemann Surfaces (The Classical Case)

An important aspect of the study of Riemann surfaces is the definition of meromorphic functions on these surfaces. With these functions comes a wealth of understanding about the nature of these mathematical objects. The idea of a single-valued function is an important concept in complex analysis. (Indeed, the property of being single-valued is an intrinsic part of the modern definition of a function.) However, many of the functions encountered in complex analysis are multi-valued. By this is meant:

Definition 1. A function $f(z)$ is said to be multi-valued if for some z, $f(z)$ corresponds to more than one distinct value in the image space of f.

For example, if we consider $f:\mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^{1/2}$, then for any $z \in \mathbb{C}-\{0\}$, $f(z) = \sqrt{z}$ takes on potentially two different values in the image space, $\pm\sqrt{z}$. Every nonzero complex number z has two complex square roots. Since most of the theory behind (elementary) complex analysis is grounded in the single-valued nature of a function on its domain, where the function assumes but one image value for every z in its domain, there developed the need to find an appropriate space in which to define such functions. During the early 1850's, the renowned mathematician Georg Friedrich Bernhard Riemann realized that one could construct a multi-layered surface on which a multi-valued function of a complex variable could be interpreted as a single-valued function. With this came the birth of the Riemann surface as a complex analytical tool.

As an example of this construction suppose we consider the function above: $f:\mathbf{C} \to \mathbf{C}$ defined by $f(z) = z^{1/2}$. In $\mathbf{C}$, the only point at which f is single-valued is $z = 0$. At all other points, f is double-valued. In fact, for $z \neq 0$, we can write $z = re^{i\theta}$ where $r > 0$ and $0 \le \theta < 2\pi$. Then, f(z) has two values: $f(z) = z^{1/2} = (re^{i\theta})^{1/2} = \begin{cases} r^{1/2}e^{i\theta/2} \\ r^{1/2}e^{i(\theta+2\pi)/2} \end{cases}$. Now beginning with $\theta = 0$, if we let θ increase to $\theta = 2\pi$ about the origin, notice that the value of f at z goes from $f(z) = r^{1/2}$ to $f(z) = r^{1/2}e^{i\pi}$. By traversing a circle centered at the origin, we have obtained a second value of f at z. If we allow θ to increase again by 2π to $\theta = 4\pi$, we obtain our original value of f at z, i.e., $f(z) = r^{1/2}$. Thus, we must traverse a circle twice in the counter-clockwise direction to obtain the same value of f at z at which we began. On $\mathbf{C}$, f is double-valued. To avoid this multi-valued nature of f, we could restrict the domain of f by removing the nonpositive real axis: $D = \mathbf{C} - \{z \mid \text{Im } z = 0, \text{Re } z \le 0\}$. By doing this, a circle about the origin can never be completely traversed. On this new domain, we can define two branches of f which are single-valued in D: $f_1(z) = r^{1/2}\, e^{i\theta/2}$; $f_2(z) = r^{1/2}e^{(\theta+2\pi)/2}$. On the other hand, Riemann suggested that perhaps we should "trade" this "complicated" (multi-valued) function on a "simple" domain for a "simple" (single-valued) function on a "complicated" domain. To construct this new domain, he suggested that for each branch of f a copy of the domain on which the branch is single-valued be taken. "Gluing" these pieces of the domain appropriately along the removed rays, an n-layered surface is then formed where n corresponds to the number of branches of f under consideration. In this case, the resulting two-layered surface would look somewhat like:

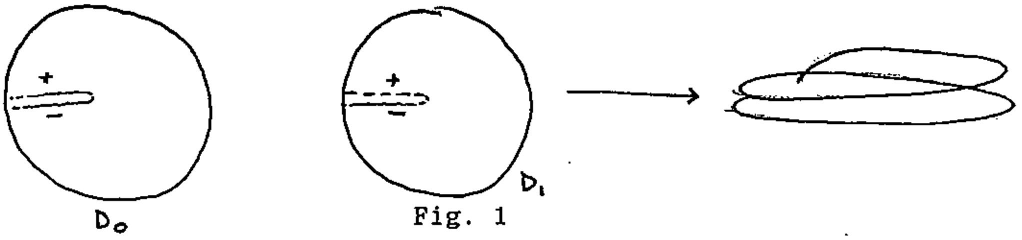


Fig. 1

This surface is formed by gluing the positive side of D_0 to the negative side of D_1, and the positive side of D_1 to the negative side of D_0.) The resulting surface, $\overline{D}$, is called the Riemann surface of $f(z) = z^{1/2}$. On this surface, D, $f(z) = \begin{cases} r^{1/2}e^{i\theta/2} & \text{if } z \in D_0 \\ r^{1/2}e^{i\theta/2+i\pi} & \text{if } z \in D_1 \end{cases}$ is single-valued as desired. If we include the point at ∞ in the domain of $z^{1/2}$, the resulting surface is more easily visualized (see Fig 2.).

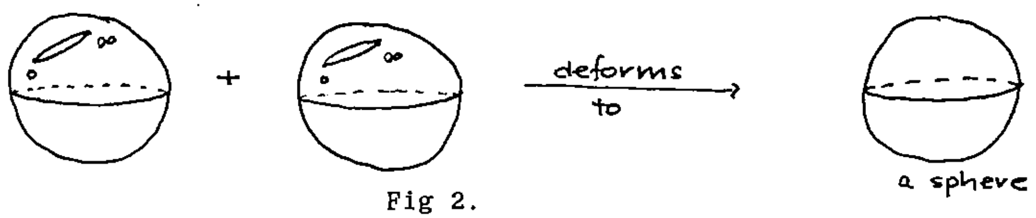


Fig 2.

Keeping this example in mind, it becomes obvious that the structure and properties of the Riemann surface are intimately tied to the nature of these multi-valued functions. Any theoretical understanding of the essence of these surfaces begins with an understanding of the behavior of such functions defined on the surface. Hence, it is the single-valued counterparts of multi-valued functions which become the objects of our interest. Notice that a function $f:S \to \mathbb{C}$ defined on a compact Riemann surface S by $f(p) = z_0 \in \mathbb{C}$ for all $p \in S$, i.e., a constant function on S, is single-valued. It is easily shown that every holomorphic function $f:S \to \mathbb{C}$ defined on a compact Riemann surface S is a constant function. Therefore, every compact Riemann surface does have single-valued constant functions associated with it. But, such functions are not very interesting from the standpoint of supplying information about S. Rather, we would like to work with non-constant meromorphic functions, i.e., functions whose singularities are no worse than a finite number of poles. Because the behavior of such functions varies on S, they may provide insight into the theoretical foundations of Riemann surfaces in a somewhat generalized form.

Naturally, we may begin to wonder if there exist non-constant meromorphic unctions on every Riemann surface. In 1851, Riemann himself supplied the nswer:

heorem 1. (The Riemann Existence Theorem) Every Riemann surface S (including ll non-compact surfaces, as well as compact) has a non-constant meromorphic unction $f \in K(S)$, where $K(S)$ is the field of all meromorphic functions on S.

his fact follows immediately from the proposition that if S is a Riemann urface and $Q,P \in S$, then there exists a meromorphic differential $\omega_{P,Q}$ on S hich has simple poles at P and Q, and no other poles, such that $\operatorname{Res}_P \omega_{P,Q} =$ 1 and $\operatorname{Res}_Q \omega_{P,Q} = -1$. In fact, such differentials exist for any $P,Q \in S$. aking the ratio of $\omega_{P,Q}$ and $\omega_{R,Q}$ where $P \neq R$, $\omega_{P,Q}/\omega_{R,Q}$, we obtain a on-constant meromorphic function on S.

Although we now know of meromorphic functions defined on a Riemann urface which possess poles as singularities and no other types of ingularity, at the moment it is beyond our means to specify at which points hese poles exist. Because we know that there do not exist nonconstant olomorphic functions on a compact Riemann surface, our choices of such poles re not completely arbitrary. Nonconstant functions on a compact Riemann urface must have at least one pole. What freedom do we enjoy in specifying he locations of poles? To facilitate the investigation of this query, we ust introduce some new notation.

If in local coordinates about $P \in S$ the analytic function f has a series xpansion $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$, $(a_n \neq 0)$, we say that n is the <u>order of</u> <u>at P</u>, written $\operatorname{ord}_P(f) = n$. Sometimes this is also referred to as the valuation of f at P. If $f \equiv 0$, we define $\mathrm{ord}_z(f) = +\infty$. Under coordinate changes in S, $\mathrm{ord}_z(f)$ is invariant. Moreover,

$$\mathrm{ord}_z(f_1 f_2)=\mathrm{ord}_z(f_1)+\mathrm{ord}_z(f_2); \quad \text{and} \quad \mathrm{ord}_z(f_1+f_2) \geq \min\{\mathrm{ord}_z(f_1),\mathrm{ord}_z(f_2)\}.$$

Likewise, for an analytic differential ω, which appears locally at $P \in S$ as $\omega = (a_n z^n + a_{n+1} z^{n+1} + \ldots)dz$; $a_n \neq 0$, we say the order of ω at P is n, written $\mathrm{ord}_z(\omega) = n$.

Our goal is to specify the locations $P_1,\ldots,P_k$ and associated orders $n_1,\ldots,n_k$ of poles for nonconstant meromorphic functions in S. To help "keep track" of these points, we define:

Definition 2. A divisor D is a formal expression of the form $D = \sum_{i=1}^{k} n_i P_i = n_1 P_1 + \ldots + n_k P_k$. (We adopt this notation to emphasize the fact that this sum does not refer to addition and scalar multiplication in $\mathbb{C}$.) The order of D at P_i is n_i. The degree of D is the integer $\sum_{i=1}^{k} n_i$.

Because a meromorphic function on a compact Riemann surface S with $f \not\equiv 0$ has only finitely many zeros and poles, we can discuss what is meant by the divisor of f. Since f has only a finite number of zeros $P_1,\ldots,P_\ell$ with associated multiplicities $n_1,\ldots n_\ell$, we define the divisor of zeros of f to be $(f)_0 = \sum_{i=1}^{\ell} n_i P_i$. Likewise, for the finite number of poles of f, $Q_1,\ldots,Q_k$ with respective orders $m_1,\ldots,m_k$, we define the divisor of poles of f to be $(f)_\infty = \sum_{i=1}^{k} m_j Q_j$. Then, the divisor of f is given by $(f) = (f)_0 - (f)_\infty = \sum_{j=1}^{\ell} n_i P_i - \sum_{j=1}^{k} m_j Q_j$. With this, then, we have found a compact manner in which o specify the poles and zeros of a function. A brief glance at this simple xpression indicates to us exactly where the singularities and zeros of f fall n S. Later, such information will prove very helpful in our study of the ature of such functions on a given surface S.

xample 1. On the compact Riemann sphere, Σ, one example of a meromorphic unction on Σ is:

$$f(z) = \frac{z^2 + z+1}{z^3-3z^2} .$$

wo of the zeros of f occur where the numerator is zero:

$$z^2+z+1=0; \quad (z-(-1/2+\sqrt{3}/2i))(z-(1/2-\sqrt{3}/2i))=0; \quad z=-1/2+\sqrt{3}/2i, \ -1/2-\sqrt{3}/2i.$$

lso, the poles of f occur where $z^3 - 3z^2 = 0$; $z^2(z-3) = 0$. The poles are: $= 0$ (mult 2), $z=3$ (simple pole). What happens at ∞? Suppose we change oordinates to $w = 1/z$. Then,

$$f(w) = \left[\frac{(1/w^2) + (1/w) + 1}{(1/w)^3 - 3(1/w)^2}\right] = \frac{w + w^2 + w^3}{1 - 3w} .$$

otice that at $w = 0$, $f(w)$ has a simple zero. This implies that at $z = \infty$, (z) also has a simple zero. Using all this information, we obtain:

$$(f)_0 = P_{-1/2+\sqrt{3}/2i} + P_{-1/2-\sqrt{3}/2i} + P_\infty .$$

$$(f)_\infty = 2\cdot P_0 + P_3 .$$

Thus, $(f) = (f)_0 - (f)_\infty = P_{-1/2+\sqrt{3}/2i} + P_{-1/2-\sqrt{3}/2i} + P_\infty - (2\cdot P_0 + P_3)$.

where for $c = 0, 3, -1/2 + \sqrt{3}/2i, -1/2 - \sqrt{3}/2i$, P_c represents the point c. The reason for such notation is that the divisor of a function is a formal sum $\Sigma n_i P_i$ in which the coefficient n_i is not to be multiplied by the point P_i to obtain a complex number. In other words, the divisor of a function is not a complex number. Rather, the divisor is a notational device which allows us to catalogue the zeros and poles and their multiplicities.

Since divisors are so convenient for a discussion of the zeros and poles of a meromorphic function, we would hope that such an entity would also exist for meromorphic differentials from which are derived the meromorphic functions. Indeed, we can also speak of the divisor of an abelian differential $\omega (\neq 0)$ on S. In fact, the definition of this divisor parallels that for the meromorphic functions. Suppose that ω is an abelian differential on S, $\omega \neq 0$, which has zeros $P_1, \ldots P_\ell$ with multiplicities $n_1, \ldots n_\ell$ and poles $Q_1, \ldots, Q_k$ of respective orders $m_1, \ldots, m_k$. Then, similar to the previous development for f, we define:

1) the divisor of zeros of ω: $(\omega)_0 = \sum_{i=1}^{\ell} n_i P_i$;

2) the divisor of poles of ω: $(\omega)_\infty = \sum_{j=1}^{k} m_j Q_j$;

3) the divisor of ω: $(\omega) = (\omega)_0 - (\omega)_\infty$.

For example, suppose $S = \Sigma$ and let ω be an abelian differential on S. Locally, we can express ω as $\omega = f(z)dz$ where f(z) is a meromorphic function on S and hence has a finite number of zeros and poles in S. From above, we can determine the divisor (f) of f. However, while the zeros and poles of f do contribute to those of ω, these do not constitute the only poles of ω. We ust also consider the behavior of dz at ∞. (The behavior of f at ∞ has lready been considered in (f)). If we change coordinates, letting $w = 1/z$, e obtain $dz = d(1/w) = -(1/w^2)dw$. At $w = 0$, corresponding to $z = \infty$, dz = (1/w) has a pole of order 2. Thus, in fact ω has an additional pole of order at ∞, as well as the zeros and poles of f. (Note: if f has a zero of some rder at ∞, the pole at ∞ may indeed be removable. On the other hand, if f as a pole at ∞, the order of the pole at ∞ of w may be greater than 2.) In ny case, we get $(\omega) = (f) - 2P_\infty$.

Before proceeding any further with a discussion of divisors as they elate to functions and differentials on S, a little more information about eneral divisors must be presented. Given any two divisors $D_1 = \sum_{i=1}^{k} n_i P_i$ and $_2 = \sum_{j=1}^{\ell} m_j Q_j$, we have the following:

<u>acts</u>:

a) The <u>sum</u> of two divisors is defined by: $D_1+D_2 = \sum_{i=1}^{k} n_i P_i + \sum_{j=1}^{\ell} m_j Q_j$. oreover, we specify that $D_1 + D_2 = D_2 + D_1$, making this operation ommutative.

b) If $P_i = Q_j$ for some i,j, then $n_i P_i + m_j Q_j = n_i P_i + m_j P_i = (n_i + m_j)P_i$.

(Thus, in the previous definition of (ω) for $S = \Sigma$, if (f) has a zero of ultiplicity n at ∞, it has a term nP_∞. Then, $(\omega) = (f)-2P_\infty = ((f)-nP_\infty) +$ $n-2)P_\infty$. If n=2, the zero and pole at ∞ both cancel. However, if $n > 2$, a ero at ∞ remains of lesser multiplicity while the pole disappears. Finally, f $n < 2$, the pole at ∞ is reduced in order and the zero at ∞ is removed.)

c) If $n_i=0$ for all i, we define $D_1=0$. Then, $\deg(D_1) = 0$.

d) $-D_1 = -\sum_{i=1}^{k} n_i P_i$ which is the same as $0-D_1 = \Sigma\, 0R_j - \Sigma n_i P_i = -\Sigma n_i P_i$.

e) This implies that: $D_1 - D_2 = \sum_{i=1}^{k} n_i P_i - \sum_{j=1}^{\ell} m_j Q_j$.

f) $\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2)$

g) $\deg(D_1 - D_2) = \deg(D_1) - \deg(D_2)$

h) We say that a divisor $D_1 = \sum_{i=1}^{k} n_i P_i$ is _effective_ (or _integral_) if $n_i \geq 0$ for all $i=1,\dots,k$, and we write $D_1 \geq 0$.

i) For two meromorphic functions f_1 and f_2 on a Riemann surface S, $(f_1)+(f_2) = (f_1 f_2)$.

j) If f is a meromorphic function ($\neq 0$) on S and ω is an abelian differential ($\neq 0$) on S, $(f\omega) = (f) + (\omega)$.

k) If $f \equiv c$ (a nonzero constant), then $(f) = 0$.

Considering the above discussion, and recalling a few facts from abstract algebra, it becomes obvious that the set of all divisors Div(S) forms a commutative group.

On a compact Riemann surface S, it is known that a meromorphic function defined on S has the same number of zeros as it has poles, counting multiplicities. Hence, if $(f) = (f)_0 - (f)_\infty = \sum_{i=1}^{k} n_i P_i - \sum_{j=1}^{\ell} m_j Q_j$, then $\sum_{i=1}^{k} n_i = \sum_{j=1}^{\ell} m_j$. Therefore, $\deg(f) = \sum_{i=1}^{k} n_i - \sum_{j=1}^{\ell} m_j = 0$ for any meromorphic function f on S. Obviously not every divisor is a divisor of a meromorphic function on S. Specifically, such divisors must be of degree zero. Suppose we consider the subset $\{(f)\}$ of Div(S) consisting of the divisors of meromorphic functions of S, and the subset $\mathrm{Div}^0(S) = \{D \mid \deg D = 0\}$. Then, $\{(f)\}$ is a subset of $\mathrm{Div}^0(S)$. Moreover,

<u>Proposition 1</u>: The subsets $\{(f)\}$ and $Div^0(S)$ of the group Div(S) of all divisors of S both form subgroups of Div(S).

Looking back on all that has come to pass thus far, given any meromorphic function or differential on S, we now have a way of neatly organizing the zeros and poles of these entities, the points of interest in a discussion of functions on a Riemann surface. While this is quite useful in itself, we still have not reached the goal of being able to determine when a function will exist on a Riemann surface which possesses zeros or poles at the points of our specification, or if any such functions can in fact be found. In particular, we would like to discover if, given any divisor D, there exist functions defined on S whose divisors satisfy $(f) + D \geq 0$. It is in this case that f will possess zeros at least at the points where the coefficient of D is < 0 and poles at most at the points where the coefficient of D is > 0.

<u>Definition 3</u>: Let $D = \Sigma n_i P_i$, $P_i \in S$ be any arbitrary divisor. We define $L(D) = \{f \in K(S) | (f) + D \geq 0\} \cup \{f = 0\}$, the set of all meromorphic functions on S whose divisors are "bounded by D".

Notice that if D is an effective divisor, $(f) + D$ is effective (≥ 0) if and only if D "cancels" the poles of f, i.e., if and only if f has poles of orders $\leq n_i$ at the points P_i and no other singularities. In this case, L(D) is exactly the set of meromorphic functions with poles bounded by D. Moreover:

<u>Proposition 2</u>: Let S be any compact Riemann surface and let $D = \Sigma m_j Q_j$ be an effective divisor on S. Then L(D) is a vector space with field of scalars $\mathbb{C}$.

Proof: First, notice that the set of meromorphic functions defined on a compact Riemann surface S, K(S), is a vector space under the usual operations of addition of functions and multiplication of functions by a scalar. If we consider L(D), then L(D) is a subset of K(S). To see that L(D) is a vector space itself, we need only show that L(D) is a subspace of K(S). This involves establishing that given any two $f, g \in L(D)$ and any $c \in \mathbb{C}$, then $cf+g \in L(D)$.

Consider the Laurent expansions of f and g at Q_j in D for each j. Since $f \in L(D)$, f has a pole of order $\leq m_j$ at Q_j. Thus, if z is a local coordinate s.t. $z = 0$ corresponds to Q_j, then

$$f(z) = a_{-m_j} / z^{m_j} + a_{-(m_{j-1})} / z^{m_{j-1}} + \text{higher order terms (h.o.t.)}$$

A similar situation holds for g at Q_j:

$$g(z) = b_{-m_j} / z^{m_j} + b_{-(m_{j-1})} / z^{m_{j-1}} + \text{h.o.t.}$$

Then we have:

$$(cf+g) = cf(z)+g(z) = \left[ca_{-m_j} / z^{m_j} + ca_{-(m_{j-1})} / z^{m_{j-1}} + \text{h.o.t.}\right] + \left[b_{-m_j} / z^{m_j} + b_{-(m_{j-1})} / z^{m_{j-1}} + \text{h.o.t.}\right]$$

$$= (ca_{-m_j} + b_{-m_j}) / z^{m_j} + (ca_{-(m_{j-1})} + b_{-(m_{j-1})}) / z^{m_{j-1}} + \text{h.o.t.}$$

Therefore, cf+g has a pole of order $\leq m_j$ at $z=0$, which corresponds to Q.

Since Q_j was arbitrarily chosen in D, this implies that cf+g $\in$ L(D). $\square$

On the other hand, suppose that D has some terms with negative coefficients (say $-n_i P_i$ with $n_i > 0$). Then, the defining condition $(f) + D \geq 0$ of L(D) implies that f $\in$ L(D) has zeros of order at least n_i at P_i. Also, the poles of f are bounded by the terms with positive coefficients in D.

Knowing that L(D) is a vector space, we can discuss the notion of the dimension of L(D).

Definition: With notation as above, we denote by dim(L(D)) the <u>dimension</u> of the vector space L(D). This refers to the number of linearly independent functions in L(D) over the complex numbers $\mathbb{C}$.

Example 2: Suppose D = 0. Then, f $\in$ L(D) would have to satisfy $(f) + 0 = (f) \geq 0$. In other words, (f) would have to be an effective divisor, which implies that f would possess possibly some zeros, but no poles. However, we saw that on a compact Riemann surface, such a function is just a constant function. Therefore, L(D) is the vector space of complex numbers, L(0) = Span{1}, and dim(L(0)) = 1.

Moreover, if D is a divisor s.t. deg(D) < 0, this implies that the sum of the absolute value of the coefficients of the negative terms exceeds that of the positive terms. In order for a function f to be an element of L(D), then, f would be required to have more zeros than poles (counting multiplicities). Saying this slightly differently, deg((f)) would have to exceed 0. But, deg((f))=0 since f is defined on S. Thus, for a divisor D s.t. deg(D) < 0, L(D) must contain no functions, i.e., L(D) = Φ.

Because a divisor of a meromorphic differential on S can be specified, it ould seem logical that a space similar to L(D) would exist for differentials.

n fact, we have:

efinition 4: Let $D = \sum_{i=1}^{k} n_i P_i$, $P_i \in S$ be an arbitrary divisor on S. We define

(D) = {meromorphic differentials ω on S | $(\omega) + D \geq 0$}.

roposition 3: If $D = \sum n_i P_i$ is an effective divisor on S, then $\Omega(D)$ is a

ector space over the field of scalars $\mathbb{C}$.

roof: The proof of this proposition is analogous to the proof given for

roposition 2. □

efinition 5: By the index of speciality of a divisor D, denoted i[D], we mean

he dimension of the space $\Omega(-D)$.

Recall that in the case of D = 0, L(0) = Span{1} and dim(L(0)) = 1. In

his situation, then, what is $\Omega(0)$ and $\dim(\Omega(0))$? Before this can be answered

n its entirety, we need the following:

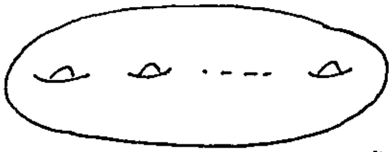
heorem 2: Every compact orientable surface (e.g. a Riemann surface) is

epresented by a polygon with edge symbol either:

1) empty, g = 0; or

2) $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$.

he number g is known as the genus of the surface.



"g-holed torus"

Fig. 3

With this definition of genus, we have:

Proposition 4: $\Omega(0) = \mathcal{H}^1(S)$, the space of holomorphic abelian differentials, and thus $i[0] = g$, the genus of S.

Proof: From the definition, we have $\Omega(0) = \{\omega | \omega$ is an abelian differential, $(\omega) + 0 = (\omega) \geq 0\}$. Thus, $\Omega(0)$ consists of the abelian differentials with $(\omega) \geq 0$, i.e., whose divisor is effective. However, this implies that $\omega \in \Omega(0)$ has possibly zeros, but no poles. Therefore, ω is a holomorphic differential and $\Omega(0) \subseteq \mathcal{H}^1(S)$. On the other hand, if $\omega \in \mathcal{H}^1(S)$, then ω has no poles and (ω) is effective. Thus, $\omega \in \Omega(0)$ and $\mathcal{H}^1(S) \subseteq \Omega(0)$. Since we have both inclusions, we have equality: $\Omega(0) = \mathcal{H}^1(S)$. Moreover, since $\dim \mathcal{H}^1(S)=g$ ([12,III.2.7]) this implies $\dim(\Omega(0)) = i[0] = g$. □

There is an interesting and quite useful relationship which exists between the dimensions of these two spaces:

Theorem 3: If ω is any abelian differential, $\omega \neq 0$, then $i[D] = \dim(L((\omega)-D))$ for any divisor D.

Proof: We begin by constructing two isomorphisms which are inverses of each other.

If $f \in L((\omega) - D)$, then $(f) + (\omega) - D \geq 0$, i.e. $(f\omega) - D \geq 0$. This implies that $f\omega \in \Omega(-D)$. Conversely, if ω is a non-zero differential of the first kind on S, and η is a meromorphic differential in $\Omega(-D)$, then $\eta/\omega = f$ is an element of K(S) and $(f) + (\omega) - D = (\eta) - (\omega) + (\omega) - D = (\eta) - D \geq 0$. Hence, $f \in L((\omega) - D)$. Thus, we have constructed two isomorphisms:

$$1)\quad \Omega(-D) \longrightarrow L((\omega) - D)$$

$$\eta \longrightarrow \eta/\omega = f$$

$$2)\quad L((\omega) - D) \longrightarrow \Omega(-D)$$

$$f \longrightarrow f\omega$$

which are inverses of each other. Therefore, $\Omega(-D)$ is isomorphic to $L((\omega)-D)$ which implies $i[D] = \dim_{\mathbb{C}}\Omega(-D) = \dim(L((\omega) - D)$ as desired. □

In specifying the nature of the vector space $L(D)$ and its dimension, where $D = \Sigma\, n_i P_i$ is an effective divisor of our own choosing, we have essentially expressed the expectation of being able to discover the existence of nonconstant functions in $K(S)$ possessing poles at P_i with multiplicities $\leq n_i$ for each i. The essence of this lies in the ability to compute the dimension of $L(D)$. If $\dim(L(D)) = 0$ or 1 and D is effective, then $L(D)$ is either empty or solely consists of the constant functions. Hence, we would like to establish a simple procedure for determining the dimension of $L(D)$. Then if we find that for our chosen D, $\dim(L(D)) \geq 2$, we will have succeeded in our task.

Armed with an understanding of the terminology presented thus far, we are now in the position to state "one of the cornerstones of the theory of compact Riemann surfaces," ([8, p. 67]) which was supplied by Riemann and his pupil Roch during the 1860's.

Theorem 4: (Riemann-Roch) Let S be a compact Riemann surface of genus g. Given a divisor D, then $\dim(L(D)) = \deg(D) + i[D] - g + 1$.

Proof: For a detailed proof, I refer the reader [12, pp. 264-269]. □

In the previous theorem it was established that for any abelian differential $\omega(\neq 0)$, $i[D] = \dim L((\omega)-D)$ given a divisor D. Using this fact, the result of the Riemann-Roch theorem can also be stated as:

$$\dim(L(D)) = \deg(D) + \dim(L((\omega)-D)) - g + 1$$

where ω is any abelian differential on S.

Hinting at its strength and importance, there are a number of immediate consequences which grow from both the statement and the proof of the Riemann-Roch theorem. As an intermediary step in the verification procedure, an interesting relation emerges, the so-called <u>Riemann Inequality</u>: $\dim(L(D)) \geq \deg(D) - g + 1$ where D is an effective divisor. This indicates that if $D = \sum_{i=1}^{k} n_i P_i$, the number of linearly independent meromorphic functions defined on S with poles of order at most n_i at the k distinct points P_i is at least $\sum_{i=1}^{k} n_i - g + 1$. We may wonder for which D there are exactly $\deg(D) - g + 1$ such functions on S. To answer this question we need the following result:

<u>Theorem 5</u>: For any abelian differential ω, $\deg[(\omega)] = 2g-2$.

<u>Proof</u>: If $g = 0$, consider the differential dz expressed in the local coordinate z. In the affine plane, dz is regular. Suppose we consider $z = \infty$. If we change coordinates to $w = 1/z$, this corresponds to $w = 0$. In this new coordinate system we have $dz = d(1/w)$ which has a double pole at ∞. Hence, dz has a double pole at ∞ and $\deg((dz)) = -2$. Recall that any other differential on a surface of genus 0 is of the form $\omega = f(z)dz$ where $f \in K(S)$. Then, $(\omega) = (f) + (dz)$. But, because $f \in K(S)$ and the genus of S is 0, we know that

f) = 0. Hence, $(\dot{\omega}) = (dz)$ and $\deg((\dot{\omega})) = \deg((dz))$ for any abelian differential ω on S, which implies $\deg((\omega)) = -2$ as desired.

Now assume $g > 0$. Then, the space of holomorphic differentials, $\Omega(0)$ has positive dimension. Let $\pi \in \Omega(0)$. By the Riemann-Roch theorem we have:

(i) $$\dim(L((\pi))) - i[(\pi)] = \deg((\pi)) - g + 1 .$$

But, recall that, taking $\omega = \pi$, $i[(\pi)] = \dim L((\omega)-(\pi)) = \dim L(0) = 1$ for some abelian differential $\omega(\neq 0)$. Also, this same theorem tells us that $\dim(L((\pi))) = i[0] = g$. Therefore, (i) becomes $g-1 = \deg(\pi)-g+1$, i.e., $\deg(\pi) = 2g-2$. □

Having established this, we now can show that:

Proposition 5: If $\deg(D) > 2g-2$, then $\dim(L(D)) = \deg(D) - g + 1$.

Proof: Notice that if $\deg(D) > 2g-2$, then $i[D] = 0$. In order for a differential ω to be in $\Omega(-D)$, it must be that $(\omega) - D \geq 0$, which implies that $\deg(\omega) \geq \deg(D) > 2g-2$. But, the above theorem states that $\deg(\omega) = 2g-2$, leading to a contradiction. Using this, then, the Riemann-Roch then reads: $\dim(L(D)) = \deg(D)-g+1$ as desired. □

Reasoning along the same line, we also have:

Theorem 6: A function cannot have a single simple pole on a surface of genus one (a torus).

Proof: From above, given any abelian differential ω on a surface S of genus 1, $\deg[(\omega)] = 2g-2 = 0$. Thus, ω must have as many zeros as poles. (Note: if ω is of the first kind, or holomorphic, it has no poles and hence no zeros.)

Suppose we specify D=P, where $P \in S$. We would like $f \in L(P)$ to have at most a simple pole at P. Notice that deg(P) = 1 > 0 = 2g-2. Therefore, dim(L(P)) = deg(P)-1+1 from the above theorem. In other words, dim(L(P)) = 1. Since the set of constant functions is a 1-dimensional subspace of L(D) for all effective divisors D, this implies that L(P) = {constant functions on S}. Thus, there are no nonconstant meromorphic functions in L(P), and no function on S has a simple pole on S. □

With this, we know that for any P on a surface of genus 1, the only functions in L(P) are the constant functions. Hence, dim(L(P)) = 1. Using the Riemann-Roch formula, dim(L(P)) = deg(P) + i[P] - g + 1, we get:

$$1 = 1 + i[P] - 1 + 1 \quad \text{which implies} \quad i[P] = 0 < 1 = g \ .$$

In fact, a similar result occurs on every compact Riemann surface of genus $g > 0$.

<u>Theorem 7</u>: If $g > 0$, there is no point P on S at which all differentials of the first kind vanish, so that $i[P] < g$. (Recall: $i[P] = \dim(\Omega(-P))$.

<u>Proof</u>: Suppose on the contrary that for all $\omega \in \mathcal{H}^1(S)$, ω does vanish at P. Then, ω is a holomorphic abelian differential and, as in Proposition 4, $\Omega(-P) = \mathcal{H}^1(S)$. So, i[P] = g. By the Riemann-Roch theorem, this implies: dim(L(P)) = 1 + g − g + 1 = 2 and there exists a nonconstant meromorphic function f on S with a simple pole at P. Thus, f assumes every complex number as a value exactly once on S. (It can be shown ([12, p. 176]) that a meromorphic function assumes every value the same number of times on a compact

iemann surface.) As a result, f defines a one-to-one holomorphic mapping f:S Σ, which indicates that the genus of S is the same as that of the sphere, .e., g=0. But, this is a contradiction, and the theorem therefore follows. □

By the Riemann Inequality, we saw that for an effective divisor D, im(L(D)) $\geq$ deg(D) - g + 1, giving dim(L(D)) a lower bound. In fact, there is nother relation which exists which indicates an upper bound for the size of he space L(D).

<u>heorem 8</u>: If D is an effective divisor, then dim(L(D)) $\leq$ deg(D)+1 with quality holding if and only if g=0, the only case in which i[D] = g(=0).

On numerous previous occasions, we have encountered the situation in hich i[D] = 0, where D is effective. In these cases, under the Riemann-Roch heorem, dim(L(D)) = deg(D)-g+1. From this, we may begin to wonder exactly hat D looks like. Let's investigate.

Since D is effective, we know that dim(L(D)) $\geq$ 1, so that the iemann-Roch formula reads: (1) deg(D) + i[D] - g $\geq$ 0. Suppose we let D = $_1+\ldots+P_n$ for distinct points of S, $P_1,\ldots,P_n$. Then, deg(D) = n and (1) mplies: (2) i[D] $\geq$ g-n. Notice that if n > g, i[D] must be 0. So, the only nteresting case in this situation is where g $\geq$ n. Assuming this, consider [P_1]. From Theorem 6 above, we saw that i[P_1] $\leq$ g-1. This coupled with (2) [P_1] $\geq$ g-1 implies i[P_1] = g-1. Suppose now D = P_1 + P_2. We know $\Omega(-P_1-P_2)$ $\Omega(-P_1)$ and i[P_1+P_2] $\leq$ i[P_1] = g-1. Moreover, if g $\geq$ 2, then there exists at east one differential φ in $\Omega(-P_1)$ which is not identically 0. Let

$_2$ be a point where $\mathcal{F}$ does not vanish. We now have $i[P_1+P_2] \leq g-2$, which together with (2) implies $i[P_1+P_2] = g-2$. Continuing in this same vein, for each $n \leq g$, there exist n distinct points $P_1,\ldots,P_n$ on S such that $i[P_1+\ldots+P_n] = g-n$. In particular, if $n = g$, there are g distinct points on S such that $i[P_1+\ldots+P_g] = 0$. Using the Riemann-Roch formula,

$$\begin{aligned}\dim(L(P_1+\ldots+P_g)) &= \deg(P_1+\ldots+P_g) + i[P_1+\ldots+P_g]-g + 1\\ &= g + 0 - g + 1\\ &= 1 .\end{aligned}$$

And so, we have established,

<u>Theorem 9</u>: It is possible to find g distinct points on a surface S of genus g such that there does not exist any nonconstant meromorphic function on S whose only singularities are poles of order at most 1 at the points $P_1,\ldots,P_g$.

Throughout the previous proof, we considered only the situation in which $\deg(D) \leq g$. On the other hand, suppose $\deg(D) > g$. By the Riemann inequality, then, $\dim(L(D)) \geq \deg(D)-g + 1 > g-g + 1 = 1$. Hence, $\dim(L(D)) \geq 2$. As a result:

<u>Theorem 10</u>: When $\deg(D) > g$, there do exist nonconstant meromorphic functions in $L(D)$.

We have seen that there are g points on a surface S of genus g such that there are no nonconstant meromorphic functions on S with at most poles of order 1 at those points. Suppose instead we begin with a single point P on S and ask if there are nonconstant meromorphic functions such that one of these functions has a pole of order n at P and no other singularities. Are there certain integers n_i such that there does not exist a nonconstant meromorphic function in K(S) possessing as its only singularity a pole of order n_i at P? The answer to this query lies in the following theorem introduced by Weierstrass during the late 1860's.

Theorem 11: (The Weierstrass "Gap" Theorem). Let S have positive genus g, and let $P \in S$ be arbitrary. There are precisely g integers $1=n_1 < n_2 < \ldots < n_g < 2g$ such that there does not exist a function $f \in K(S)$ holomorphic on $S-\{P\}$ with a pole of order n_j at P. (In other words, no meromorphic function exists having a pole of order n_j at P as its only singularity.)

Before proving this theorem, there are a few remarks to be made about these integers. The numbers $n_1,\ldots,n_g$ are called the Weierstrass gaps (or just the "gaps") at the point P. The complement of this set in Z, $Z-\{n_1,\ldots,n_g\} = T$ is the set of "non-gaps" at P. It is clear that T is a commutative semi-group under addition of integers. Since 0 is not a "gap," $0 \in T$ and hence T has an identity element. Moreover, the sum of "non-gaps" is a "nongap." To see this, let α_1 and α_2 be two "nongaps" of P. Then, there exist two functions f_1 and f_2 in K(S) such that f_i has as its only singularity a pole of order α_i at P. Then, f_1f_2 has a pole of order $\alpha_1 + \alpha_2$ at P. Hence $\alpha_1 + \alpha_2$ is again a "nongap."

Consider the integers $\{1,\ldots,2g\}$. Within this set, there are g "gaps." This implies that there are g "non-gaps" in this set also. Notice that 1 is always a "gap" on a compact Riemann surface, and 2g is always a "non-gap."

First, to see that 2g is always a "non-gap," consider: From the Riemann-Roch theorem we have:

$$\dim(L(2gP)) = \deg(2gP) - g + 1 + \dim L((\omega) - 2gP) .$$

Now, $\deg(\omega) = 2g-2$ for any abelian differential ω. Hence, ω can have at most a zero of order $2g-2$ at P. In other words, it cannot have a zero of order 2g at P. Hence, $\dim L((\omega)-2gP) = 0$ and we have:

$$\dim(L(2gP)) = 2g - g + 1 = g + 1.$$

If $g \geq 1$, then $\dim(L(2gP)) \geq 2$ and there exists a meromorphic function on S with a pole of order 2g at P. Thus, 2g is a "non-gap."

Now the claim is that 1 is always a gap. To see this, recall from the discussion preceeding Theorem 8, we found that $i[P] = g-1$. So, by the Riemann-Roch formula, $\dim(L(P)) = 1 - g + 1 + g - 1 = 1$. Therefore, the only functions in L(P) are the constant functions. In other words, there are no nonconstant meromorphic functions on S with a simple pole at P as its only singularity, and hence $n=1$ is a "gap" for any P. The g "non-gaps" in this set are known as the <u>first g "non-gaps"</u> in the semi-group T of non-gaps.

Remember that on the Riemann sphere, whose genus is $g = 0$, one can always find a function with a simple pole at a specific point. Hence, there are no "gaps" on Σ. In this case, the results of the Weierstrass "Gap" theorem follow trivially.

<u>Proof of the Weierstrass "gap" Theorem:</u> Let P be an arbitrary point on the Riemann surface S. For D = P, we have seen that $\dim(L(P)) = 1 = \dim(L(0))$, so

j_1 is a "gap".

In the general case, consider the transition from $D = (n-1)P$ to $D = nP$. First, notice that $\dim(L((n-1)P)) = (n-1)+i[(n-1)P]-g+1 = n-g+i[(n-1)P]$ while $\dim(L(nP)) = n+i[nP]-g+1$. If $i[D]$ remains unchanged under this transition, i.e., if $i[(n-1)P] = i[nP]$, then $\dim(L(nP)) = n+i[(n-1)P]-g+1 = \dim(L((n-1)P)+1$. Hence, with this, there is a function f in $L(nP)$, but not in $L((n-1)P)$. In other words, there exists a function on S which has as its only singularity a pole of order n at P. On the other hand, if under this transition $i[nP] = i[(n-1)P]-1$, then $\dim(L(nP)) = n+i[(n-1)P]-1-g+1 = n+i[(n-1)P]-g = \dim(L(n-1)P))$. Thus, here there is no function which is regular on $S-\{P\}$ and possesses a pole of order n at P.

We may ask: How often is a new function added in going from $L((n-1)P)$ to $L(nP)$? In other words, how often does $i[nP]$ remain invariant under the transition from $D = (n-1)P$ to $D = nP$? The answer comes from a consideration of the fact that $i[0] = g$, while $i[P+...+P] = i[2gP] = 0$. From this, we see that $i[nP]$ changes (decreasing in value by one each time) exactly g times. Hence, there are exactly g values of n for which no meromorphic function exists which possesses as its only singularity a pole at P of order n. □

Above, we were dealing with a single point on S. It is interesting to note that there is a more general theorem, called the Noether "Gap" Theorem, which indicates that the "gaps" can occur at n_i distinct points on S, and not solely at P. In fact, the Weierstrass "Gap" theorem is just a specific case of this more general theorem. (For a discussion of the Noether "Gap" theorem, the reader may consult [5, p. 79].)

It follows from the proof of this theorem that $j \geq 1$ is a "gap" at $P \in S$

if and only if $\dim(L(jP)) - \dim(L((j-1)P)) = 0$ if and only if $i[(j-1)P] - i[jP] = 1$, i.e., if and only if there exists on S an abelian differential of the first kind with a zero of order j-1 at P. Thus, for S compact, the possible orders of abelian differentials of the first kind at P are precisely:

$$0=n_1-1 < n_2 - 1 <\ldots< n_g-1 \leq 2g-2$$

where the n_j's are the gaps at P. In fact, this says that given a point P on a compact Riemann surface S of genus $g > 0$, then there exists an abelian differential ω of the first kind ($\omega \in \mathcal{H}^1(S)$) that does not vanish at P. Considering this closely, it becomes obvious that this is just a restatement of Theorem 7.

At this point, it is appropriate to enumerate a number of propositions, stated in terms of the "non-gaps," which allow us to say a bit more about the nature of the "gaps."

Throughout the following propositions, let $P \in S$ be arbitrary and let $1 < \alpha_1 <\cdots< \alpha_g = 2g$ be the first g "non-gaps" at P.

<u>Proposition 6</u>: For each integer j, $0 < j < g$, we have $\alpha_j + \alpha_{g-j} \geq 2g$.

<u>Proof</u>: Suppose that $\alpha_j + \alpha_{g-j} < 2g$. Then, since the α_j are ordered, if $k < j$, then $\alpha_k + \alpha_{g-j} < 2g$ also. So, we have:

$$\begin{aligned} \alpha_1 + \alpha_{g-j} &< 2g \\ \alpha_2 + \alpha_{g-j} &< 2g \\ &\vdots \\ \alpha_j + \alpha_{g-j} &< 2g\,. \end{aligned}$$

otice that $\alpha_k + \alpha_{g-j}$ is a nongap between α_{g-j} and α_g. This is because the um of nongaps is a nongap and $\alpha_j > 0$ for all j by assumption. From the above ystem of inequalities, we have then that there are j nongaps between α_{g-j} and g. Hence, we have a total of (# of nongaps from α_1 to α_{g-j}) + (# nongaps rom α_{g-j} to α_g) + (1 nongap at α_g) = g − j + j + 1 = g + 1 nongaps, ontradicting the fact that there are only g nongaps from 1 to 2g. Thus, $_j+\alpha_{g-j} \geq 2g$. □

Knowing that $\alpha_1 = 2$ gives us a way to explicitly state the first g non-gaps" at P.

<u>roposition 7</u>: If $\alpha_1 = 2$, then $\alpha_j = 2j$ and $\alpha_j + \alpha_{g-j} = 2g$ for all j, $< j < g$.

<u>roof</u>: If $\alpha_1 = 2$, then 2,4,6,...,2g are g "non-gaps" $\leq$ 2g, and hence these are ll the "nongaps" $\leq$ 2g. To see this, if 2 is a nongap of P, then there exists meromorphic function which has a pole of order 2 at P. But, then $f(z))^2$ has a pole of order 4, and therefore 4 is a nongap. Continuing, in eneral $(f(z))^n$ has a pole of order 2n, so 2n is a nongap for $n \in Z$. oreover, $\alpha_j + \alpha_{g-j} = 2j + 2(g-j) = 2g$ for $0 < j < g$. □

What happens if $\alpha_1 > 2$?

<u>roposition 8</u>: If $\alpha_1 > 2$, then for some j with $0 < j < g$, we have $\alpha_j+\alpha_{g-j}>2g$.

<u>roof</u>: From Proposition 6 we know that $\alpha_j + \alpha_{g-j} \geq 2g$ for all j, $0 < j < g$.

Suppose for some j, $0 < j < g$, $\ddot{\alpha}_j + \ddot{\alpha}_{g-j} = 2g$. For any $q \in \mathbb{R}$, let $[q]$ be the largest integer $\leq q$. Then, $\alpha_1, 2\alpha_1, \ldots, [2g/\alpha_1]\alpha_1$ are "non-gaps" $\leq 2g$. (In the proof of Proposition 7 we saw that integer multiples of "non-gaps" are "non-gaps.") But, since $\alpha_1 > 2$, i.e., $\alpha_1 \geq 3$, $[2g/\alpha_1] \leq 2g/3$. So the above account for at most $2g/3 < g$ "non-gaps" and there must be another "nongap" $\leq 2g$. Let α be the first "nongap" not appearing in our list. For some integer r, $1 \leq r \leq [2g/\alpha_1] < g$ we must have $r\alpha_1 < \alpha < (r+1)\alpha_1$. Therefore, the first r+1 non-gaps are α_1, $\alpha_2 = 2\alpha_1, \ldots, \alpha_r = r\alpha_1, \alpha_{r+1} = \alpha$. By our assumption that $\alpha_j + \alpha_{g-j} = 2g$, this gives:

$$\alpha_{g-1} = 2g-\alpha_1, \ldots, \alpha_{g-r} = 2g - r\alpha_1,\ \alpha_{g-(r+1)} = 2g-\alpha\ .$$

For $\alpha_{g-1}, \ldots, \alpha_{g-(r+1)}$, these are non-gaps $\leq 2g$ and $\geq \alpha_{g-(r+1)}$. Moreover, $\alpha > k\alpha_1$, for all k, $0 < k \leq r$ since α was chosen to be the first "non-gap" in our enumeration. Hence, $2g-\alpha < 2g-k\alpha_1$ which implies $\alpha_{g-(r+1)} < \alpha_{g-k}$ for all k, $0 < k \leq r$. In addition, we can say that $\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}, \alpha_{g-(r+1)}, \ldots, \alpha_{g-1}$ are <u>all</u> the "nongaps" $\geq \alpha_{g-(r+1)}$ and $\leq 2g$. To see this, suppose β was another nongap in the range $\alpha_{g-(r+1)}$ to 2g. Then, $2g-\beta$ is a "non-gap." But, $2g-\beta$ is in the range $\ddot{\alpha}_1$ to $\ddot{\alpha}_{r+1} = \ddot{\alpha}$, and $\ddot{\alpha}$ was chosen as the first "nongap" not appearing in our enumeration. Therefore, $2g-\beta$ cannot be a "nongap" in this range. This is a contradiction. So, there is no other "nongap" in the range $\alpha_{g-(r+1)}$ to 2g.

With this, then, it follows that

$$\ddot{\alpha}_1 + \ddot{\alpha}_{g-(r+1)} = \ddot{\alpha}_1 + 2g - \ddot{\alpha} = 2g - (\ddot{\alpha} - \ddot{\alpha}_1)\ .$$

But, $\alpha < (r+1)\alpha_1$ (from above) implies $\alpha - \alpha_1 < r\alpha_1$. So, $\alpha_1 + \alpha_{g-(r+1)} > 2g - r\alpha_1 = \alpha_{g-r}$. However, this gives a "non-gap" between $\alpha_{g-(r+1)}$ and 2g, a contradiction. So, our original assumption was false and $\alpha_j + \alpha_{g-j} > 2g$. □

Using these ideas, it is possible to establish a lower bound for the sum of the first g-1 "non-gaps" at P:

Corollary 1: We have $\sum_{j=1}^{g-1} \alpha_j \geq g(g-1)$, with equality if and only if $\alpha_1 = 2$.

Proof: Using proposition 6, summing both sides of the resulting inequality yields:

$$\sum_{j=1}^{g-1} (\alpha_j + \alpha_{g-j}) \geq \sum_{j=1}^{g-1} 2g = 2g \sum_{j=1}^{g-1} 1 = 2g(g-1) .$$

But,

$$\begin{aligned} \sum_{j=1}^{g-1} (\alpha_j + \alpha_{g-j}) &= (\alpha_1 + \alpha_{g-1}) + (\alpha_2 + \alpha_{g-2}) + \cdots + (\alpha_{g-2} + \alpha_2) + (\alpha_{g-1} + \alpha_1) \\ &= 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{g-1} \\ &= 2 \sum_{j=1}^{g-1} \alpha_j . \end{aligned}$$

Thus, $2 \sum_{j=1}^{g-1} \alpha_j \geq 2g(g-1)$, which implies $\sum_{j=1}^{g-1} \alpha_j \geq g(g-1)$.

Now, if $\alpha_1 = 2$, then from Proposition 7 we have

$$\sum_{j=1}^{g-1} (\alpha_j + \alpha_{g-j}) = 2 \sum_{j=1}^{g-1} \alpha_j = \sum_{j=1}^{g-1} 2g = 2g \sum_{j=1}^{g-1} 1 = 2g(g-1)$$

as desired. □

Notice from Proposition 8, if $\alpha_1 > 2$, the corollary reads: $\sum_{j=1}^{g-1} \alpha_j > g(g-1)$, a strict inequality.

Within limits, we have been able to describe some of the characteristics of the first g "non-gaps" at P. In fact, in conjunction with the Weierstrass "gap" theorem, we have been able to somewhat characterize the nature of the space $L(gP)$. In particular, on a compact Riemann surface S of genus g, 1 is always a gap. Hence, $\dim(L(P)) = 1$ for all $P \in S$. Moreover, if $\alpha_1 = 2$ in the series of "non-gaps" at P, then $\{1,3,5,\ldots,2g-1\}$ are the g "gaps" of S at P. From Proposition 8, if $\alpha_1,\ldots,\alpha_g$ are the first g non-gaps with $\alpha_1 = 2$ and $n_1,\ldots,n_g$ are the g gaps at P,

$$\sum_{j=1}^{g-1} (\alpha_j + n_j) = \sum_{j=1}^{2g-2} j \quad \text{which implies} \quad \sum_{j=1}^{g-1} \alpha_j + \sum_{j=1}^{g-1} n_j = (2g-2)(2g-1)/2$$

$$\text{which implies} \quad g(g-1) + \sum_{j=1}^{g-1} n_j = (2g-2)(2g-1)/2.$$

Hence, $\sum_{j=1}^{g-1} n_j = (2g-2)(2g-1)/2 - g(g-1) = (2g-1)(g-1) - g(g-1) = (g-1)^2$.

In the general case where α_1 may not be 2, this becomes

$$\sum_{j=1}^{g-1} \alpha_j + \sum_{j=1}^{g-1} n_j \geq g(g-1) + \sum_{j=1}^{g-1} n_j \quad \text{which implies}$$

$$(2g-2)(2g-1)/2 \geq g(g-1) + \sum_{j=1}^{g-1} n_j$$

$$\sum_{j=1}^{g-1} n_j \leq (2g-2)(2g-1)/2 - g(g-1) = (g-1)^2 \quad \text{for the gaps } n_j \text{ at P.}$$

The entire preceding discussion was based on the existence of meromorphic 'unctions with poles of various orders at a specific $P \in S$. Indeed, since leg(nP)=n, if $n > g$ then $\dim(L(nP)) \geq 2$ and there do exist nonconstant ıeromorphic functions whose only singularity is a pole of order at most n at ›. However, if $n = g$, there are nonconstant functions in $L(gP)$ only if $i[gP]$ › 0, which is the case when $\dim(L(gP)) \geq 2$. Under these conditions, at least ›ne of the integers 2,...,g is a "non-gap." It is natural to wonder how many ;uch

› $\in S$ exist.

'heorem 12: There are only a finite number of points P on S at which :[gP] > 0.

'**roof**:

Assume that there are an infinite number of points $\{P_n\}$ at which $i[gP_n] >$), $n = 1,2,\ldots$. These points have a limit point P_0 on S, since S is compact. ,et $z = \Phi(P)$, $\Phi(P_0) = 0$ be a local parameter about P_0. Then, if $\{\varphi_1,\ldots,\varphi_g\}$ ıre the basis differentials of $\mathcal{H}'(S)$, $\varphi_j = f_j(z)dz$ where the $f_j, j=1,\ldots,g$ are linearly independent functions about P_0. Assume that the P_n lie in a parametric neighborhood of P_0 and $\Phi(P_n) = z_n$. Since $i[gP_n] > 0$, for each z_n :here exists a $\varphi = c_1\varphi_1 + \ldots + c_g\varphi_g \in \Omega(-gP_n)$ with $\sum_{i=1}^{g} |c_i|^2 \neq 0$ and

$$c_1 f_1(z_n) + \ldots + c_g f_g(z_n) = 0$$

$$c_1 f_1'(z_n) + \ldots + c_g f_g'(z_n) = 0$$

$$\vdots$$

$$c_1 f_1^{(g-1)}(z_n) + \ldots + c_g f_g^{(g-1)}(z_n) = 0 .$$

Thus, the Wronskian

$$W_g(z) = \det\begin{bmatrix} f_1 & \cdots\cdots & f_g \\ f_1' & & f_g \\ \vdots & & \vdots \\ f_1^{(g-1)} & & f_g^{(g-1)} \end{bmatrix}$$

must vanish at the points z_n. Since $W_g(z)$ is a holomorphic function of z, and since it is = 0 at infinitely many points in a neighborhood N_{P_0} of P_0, this implies $W_g(z) \equiv 0$. But, this implies $f_1,\ldots,f_g$ are linearly dependent, contradicting the fact that $f_1,\ldots,f_g$ were chosen to be linearly independent. Therefore, there are only a finite number of points $\{P_n\}$ on S at which $i[gP_n] > 0$. □

As a consequence of this theorem, we now know that there are only a finite number of points at which there exist nonconstant meromorphic functions on S whose only singularity is that pole of order g or less. These points are called the <u>Weierstrass points</u> of S. Notice that a surface of genus g = 0 or g = 1 has no Weierstrass points. To see this, note:

(a) For g = 0, i[gP] = i[0] = g = 0. Hence, for all P, i[gP] is never more than 0. Thus, this surface has no Weierstrass points by the above theorem.

(b) In the case g = 1, for P to be a Weierstrass point, we must have: $\dim(L(gP)) = \dim(L(P)) \geq 2$. But, by Theorem 5, $\dim(L(P)) = 1$. There can be no Weierstrass points on S of genus 1.

These two cases are special. In general, for a surface of genus $g \geq 2$, it can be shown that there do indeed exist Weierstrass points. At present, it is known that different Riemann surfaces S have different "constellations" of

Weierstrass points. However, there still remain unanswered questions concerning which combinations can occur. Instead, estimates exist as to the number W of these points which exist on a Riemann surface S of genus g. These approximations take the form: $2g+2 \leq W \leq (g-1)g(g+1)$. (This will be proved later.)

As is typical of every important concept in mathematics, there is another manner in which to view Weierstrass points and their definition. However, before we begin with this second method of definition, a few more ideas concerning these points must be presented.

Let $P_1,\ldots,P_n$ be the Weierstrass points in S. By the Weierstrass "Gap" theorem, each P_i has associated with it g integers $n_{i1},\ldots,n_{ig}$ such that there does not exist a function $f \in K(S)$ holomorphic on $S-\{P_i\}$ with a pole of order n_{ij} at P_i. If these "gaps" at P_i are $\{1,3,5,\ldots,2g-1\}$, then the Weierstrass point P_i is said to be hyperelliptic. In fact, it has been proved that if all of the Weierstrass points of a surface of genus $g \geq 2$ are hyperelliptic, then the number of such points on S is $2g+2$.

As an example, we will consider a hyperelliptic curve of genus 2.

Example 3: The curve C defined by $y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ is a hyperelliptic curve of genus 2. If the a_i are all real, the picture of C in the affine plane is similar to the following:

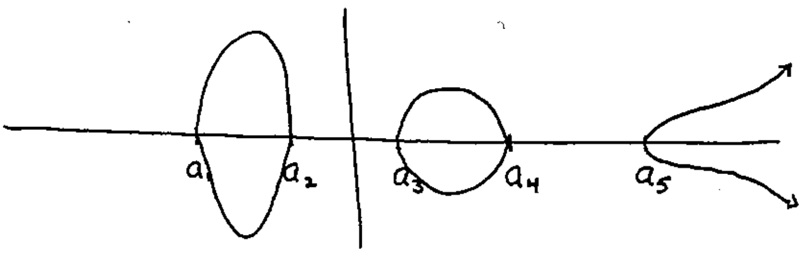


Fig. 4

In $\mathbb{P}^2$, this looks like:

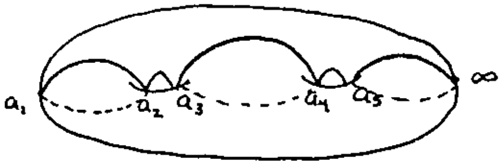


Fig. 5

I claim that on C, there are $2g + 2 = 2(2) + 2 = 6$ total Weierstrass points. Moreover, these points are $a_1, \ldots, a_5, \infty$. Since P is a Weierstrass point if and only if there exists a differential of the first kind ω with a zero of order $g = 2$ at P, to begin our investigation we must determine the nature of the space of differentials of the first kind, $\mathcal{H}^1(C)$ on C. Since dim $\mathcal{H}^1(C) = g$, we know that $\mathcal{H}^1(C)$ has dimension 2. So, we must find two linearly independent differentials of the first kind on C which will then form a basis. I claim that $\omega = dx/y$ and $\pi = xdx/y$ are two such differentials.

Consider $\omega = dx/y$. If we rewrite C as $y^2 - \Pi(x-a_i) = 0$, i.e. $f(x,y) = 0$, then $\partial f/\partial y = 2y$. Therefore, $1/y = 2/\partial f/\partial y$ and ω becomes $\omega = 2\, dx/\partial f/\partial y$. If x is a local coordinate on a portion of the curve where the tangent line is not vertical (it may be horizontal) at each point of that segment of C, then $\partial f/\partial y \neq 0$. Hence, ω has no poles there. If you notice, since $f(x,y) = 0$, we have $0 = d(f(x,y)) = \partial f/\partial x dx + \partial f/\partial y dy$. This rearranges to read: $dx/\partial f/\partial y = - dy/\partial f/\partial x$, and ω becomes $\omega = -2\, dy/\partial f/\partial x$. Then, on the portion of the curve where the tangent line is vertical, $\partial f/\partial x \neq 0$ and we can use y as a local coordinate on those portions of the curve. In that case, ω again has no poles. Since ω agrees under changes of coordinates, this all implies that ω has no poles on C in the affine plane. A similar argument shows that $\pi = xdx/y|_C$ also has no poles in the affine plane.

Thinking of C as a 2-sheeted branched covering of Σ, we change coordinates $x = 1/s$. Then, C becomes:

$$y^2 = \prod_{i=1}^{5} (1/s - a_i)$$

$$= (s \prod_{i=1}^{5} (1-a_i s))/ s^6 .$$

So, $(ys^3)^2 = s \prod_{i=1}^{5} (1-a_i s)$.

Suppose we let $w = ys^3$ so that $y = w/s^3$. At this point we want to show that $x^i dx/y|_C$ has no poles at $s = 0$. Using the above coordinates $x = 1/s$ $y = w/s^3$, $x^i/y\ dx|_C$ becomes:

$$\frac{d(1/s)/s^i}{w/s^3} = \frac{-ds/s^{i+2}}{w/s^3} = \frac{-s^{1-i}ds}{w}$$

Since $i = 0,1$, the s factor remains in the numerator, contributing a zero to the differential at $s = 0$. Also, at $s = 0$, $ds = 0$, and $w = 0$. So, the differential has a simple pole and at least a double zero at $s = 0$ from the ds term. But, the pole is cancelled by the zero and hence, the differential has no pole at $s = 0$, which corresponds to ∞.

Therefore, the differentials ω and π are of the first kind on C. Moreover, because π has a zero at $x = 0$ in local coordinates and ω does not, this implies that ω and π are in fact linearly independent. Hence, $\{\omega,\pi\}$ is a basis for $\mathfrak{H}^1(C)$. All differentials of the first kind on C look like:

$$c_1\ dx/y + c_2\ x/y\ dx = \frac{(c_1 + c_2\ x)dx}{y}$$

where $c_i \in \mathbf{C}$.

Now, a point P is a Weierstrass point on C if i[2P] > 0, i.e., if there exists an $\omega \in \mathcal{H}^1(C)$ with a double zero there. Above, I claimed that $a_1, \ldots, a_5, \infty$ were Weierstrass points of C. Suppose we consider a_i. Let $\nu = (x-a_i)dx/y$. Since at a_i the tangent line is vertical, we must change coordinates to y, $\nu = (x-a_i)\dfrac{-2dy}{\partial f/\partial x}$. From the $(x-a_i)$ factor, we get a zero of order 1 at a_i. However, in these new coordinates, dy = 0 also, contributing yet another zero at a_i. Thus ν has a double zero at a_i and thus a_i is a Weierstrass point of C.

At $P = \infty$, corresponding to s = 0, we saw that in changing coordinates $\omega = -sds/w$ and $\pi = -ds/w$. So, consider $\omega = -sds/w$. At s = 0, the s factor contributes a simple zero and the ds term a double zero. However, w also has a simple zero at s = 0. But, this cancels with only one of the zeros in the numerator, leaving ω with a double zero. Therefore, by definition, s = 0, which corresponds to $x = \infty$, is a Weierstrass point of C.

Thus, $a_1, \ldots, a_5$, ∞ are all Weierstrass points of C. In fact, these are the only such points on C. Hence, the number of Weierstrass points, W is 6 as claimed. □

In general, all surfaces of genus 1 or 2 are hyperelliptic. A surface is hyperelliptic if and only if there exists an effective divisor D of degree 2 s.t. dim(L(D)) = 2.

Having tested the waters with a concrete example, it is now appropriate to proceed with the compatible definition of Weierstrass points mentioned above. First, some basic concepts.

Let A be a finite-dimensional space of holomorphic functions on a domain $D \subset \mathbf{C}$, such that $\dim(A) = n \geq 1$. Let z be any point in D. By a <u>basis of A</u>

dapted to z we mean a basis $\{\varphi_1, \ldots, \varphi_n\}$ with $\mathrm{ord}_z\varphi_1 < \mathrm{ord}_z\varphi_2 < \ldots < \mathrm{ord}_z\varphi_n$.
o construct such a basis, let $\mu_1 = \min_{\varphi \in A}\{\mathrm{ord}_z\varphi\}$ and choose $\varphi_1 \in A$ with $\mathrm{ord}_z\varphi_1 =$ $_1$. Then, if $A_1 = \{\varphi \in A \mid \mathrm{ord}_z\varphi > \mu_1\}$, $A = \mathrm{Span}\{\varphi_1\} \oplus A_1$ and A_1 is an n-1)-dimensional subspace of A. Again, let $\mu_2 = \min_{\varphi \in A_1}\{\mathrm{ord}_z\varphi\}$ and choose $\varphi_2 \in$ $_1$ such that $\mathrm{ord}_z\varphi_2 = \mu_2$. Continuing in this vein, the desired basis will be onstructed. Since at each step any φ_i satisfying $\mathrm{ord}_z\varphi_i = \mu_i$ is chosen, this asis is not unique. However, we can create a unique basis adapted to z which will depend only on the local coordinate chosen) by multiplying the aylor expansions of the φ_i by an appropriate constant to yield (if ξ is the ocal coordinate):

$$\varphi_1 = (\xi - z)^{\mu_1} + \text{h.o.t.}$$
$$\varphi_2 = (\xi - z)^{\mu_2} + \text{h.o.t.}$$
$$\vdots$$
$$\varphi_2 = (\xi - z)^{\mu_n} + \text{h.o.t.}$$

claim that $\mu_j \geq j-1$ for all j. To see this, use a proof by induction. irst, notice that since these functions are holomorphic, $\mu_1 \geq 0$. Thus, $_1 \geq 1-1 = 0$ as desired. Now, assume this holds for j and show for j + 1. ince it holds for j, $\mu_j \geq j-1$. But, by construction of the basis, $_{j+1} > \mu_j \geq j-1$. Thus, $\mu_{j+1} \geq j = (j+1)-1$ as desired.

efinition 6: We define the weight of z with respect to A by $\gamma(z) = \sum_{=1}^{n} (\mu_j - j + 1)$. Notice that since $\mu_j \geq j-1$ for all j, $\gamma(z) \geq 0$ also.

Examples 4:

1. As an example, suppose that $\mu_1 = 1$, $\mu_2 = 2, \ldots, \mu_n = n$. Then
$$\gamma(z) = \sum_{j=1}^{n} (j)-j+1 = \sum_{j=1}^{n} 1 = n.$$
2. Suppose now that $\mu_1 = 2$, $\mu_2 = 4, \ldots, \mu_g = 2g$. These are the first g non-gaps at P, if the first nongap is 2. Then,

$$\gamma(z) = \sum_{j=1}^{g} ((2j)-j+1) = (\sum_{j=1}^{g} j)+g = g(g+1)/2 + g = g(g+3)/2 .$$

Now compare this with the situation $\mu_1 = 1$, $\mu_2 = 2, \ldots, \mu_g = g$, which from above gives $\gamma(z) = g$. Since $g < g(g+3)/2$, this implies that indeed there are integers in $\{1,\ldots,g\}$ which are not contained in $\{\mu_1,\ldots,\mu_g\}$. In short, the more positive the weight, the greater the deviation of the sequence $\{\mu_1,\ldots,\mu_g\}$ from $\{0,\ldots,g-1\}$. There is another interesting way to interpret $\gamma(z)$.

Proposition 10: Let $\{\varphi_1,\ldots,\varphi_n\}$ be any basis for A. Consider the holomorphic function (the Wronskian)

$$W_g(z) = \det\begin{bmatrix} \varphi_1(z) & \cdots & \varphi_n(z) \\ \varphi_1'(z) & \cdots & \varphi_n'(z) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(z) & \cdots & \varphi_n^{(n-1)}(z) \end{bmatrix} .$$

Then, $\gamma(z) = \mathrm{ord}_z(W_g(z))$, the order of vanishing of the Wronskian.

Proof: Notice first that a change of basis will lead to a non-zero constant multiple of W_g. To see this, recall that if $\{\psi_1,\ldots,\psi_n\}$ is a second basis for

, then $\varphi_k = \sum_{i=1}^{n} a_{ki}\varphi_i$ for each k = 1,...,n where $a_{ki} \in \mathbb{C}$. In other words,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} .$$

et A be the invertible matrix $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$. From this we get,

$$A\begin{bmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{bmatrix} = \begin{bmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{bmatrix} \text{ and in general } A\begin{bmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_n^{(k)} \end{bmatrix} = \begin{bmatrix} \varphi_1^{(k)} \\ \vdots \\ \varphi_n^{(k)} \end{bmatrix} .$$ And so,

$$\det\left[\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \vdots & & \\ \varphi_n^{(n-1)} & & \varphi_n^{(n-1)} \end{bmatrix} A^t\right] = W_g(z) .$$ Since A is invertible,

et $A^t \neq 0$.

Because of this, we may assume that the basis under consideration is the ne adapted to z. Write $W_g(z) = \det[\varphi_1 \cdots \varphi_n]$. Next, notice that et$[f\ddot{\psi}_1,\dots f\ddot{\psi}_n] = f^n \det[\ddot{\psi}_1\cdots\ddot{\psi}_n]$. This we can see as follows:

$$\det[f\psi_1\cdots f\psi_n] = \det\begin{bmatrix} f\psi_1 & \cdots\cdots & f\psi_n \\ f\psi_1'+f'\psi_1 & \cdots\cdots & f\psi_n'+f'\psi_n \\ \vdots & & \vdots \\ f\psi_1^{(n-1)}+\dots+f^{(n-1)}\psi_1 & & f\psi_n^{(n-1)}+\dots+f^{(n-1)}\psi_n \end{bmatrix}$$

$$= f\det\begin{bmatrix} \psi_1 & & \psi_n \\ f\psi_1'+f'\psi_1 & & f\psi_n'+f'\psi_n \\ \vdots & \cdots & \vdots \\ f\psi_1^{(n-1)}+\cdots+f^{(n-1)}\psi_1 & & f\psi_n^{(n-1)}+\cdots+f^{(n-1)}\psi_n \end{bmatrix}$$

$$= f\det\begin{bmatrix} \psi_1 & & \psi_n \\ f\psi_1 & & f\psi_n' \\ \vdots & \cdots & \vdots \\ f\psi_1^{(n-1)}+\cdots+f^{(n-1)}\psi_1 & & f\psi_n^{(n-1)}+\cdots+f^{(n-1)}\psi_n \end{bmatrix}$$

y adding $-f'$ times row 1 to row 2. Similarly, we can remove multiples of ψ' rom each row by adding multiples of row 1 to each.

$$= f\det\begin{bmatrix} \psi_1 & & \psi_n \\ f\psi_1' & & f\psi_n' \\ \vdots & \cdots & \vdots \\ f\psi_1^{(n-1)}+\cdots+(n-1)f^{(n-2)}\psi_1' & & f\psi_n^{(n-1)}+\cdots+(n-1)f^{(n-2)}\psi_n' \end{bmatrix}$$

$$= f^2\det\begin{bmatrix} \psi_1 & & \psi_n \\ \psi_1' & & \psi_n' \\ \vdots & \cdots & \vdots \\ f\psi_1^{(n-1)}+\cdots+(n-1)f^{(n-2)}\psi_1' & & f\psi_n^{(n-1)}+\cdots+(n-1)f^{(n-2)}\psi_n' \end{bmatrix}$$

Repeating this procedure for each row and each $\tilde{\varphi}^{(j)}(z)$, we end up with: $\varphi^{n}\det[\varphi_1 \cdots \varphi_n]$ as desired.

Using induction on n, we are now ready to prove the proposition:

(1) For n=1, $W_g(z) = \det(\varphi_1(z)) = \varphi_1(z)$. Thus, $\text{ord}_z W_g(z) = \text{ord}_z \varphi_1(z) = \mu_1 = \gamma(z)$.

(2) Assume the proposition holds for k, so that $\text{ord}_z \det[\varphi_1 \cdots \varphi_k] = \sum_{j=1}^{k} (\mu_j - j + 1)$ where $\mu_j = \text{ord}_z \varphi_j$. Consider $\det[\varphi_1, \ldots, \varphi_{k+1}] =$

$$= \det\left[\varphi_1, \varphi_1(\varphi_2/\varphi_1) \ldots, \varphi_1\left[\frac{\varphi_{k+1}}{\varphi_1}\right]\right] = \varphi_1^{k+1}\det[1, \varphi_2/\varphi_1, \ldots, \varphi_{k+1}/\varphi_1]$$

$$= \varphi_1^{k+1}\det\begin{bmatrix} 1 & \varphi_2/\varphi_1 & \\ 0 & (\varphi_2/\varphi_1)' & \cdots \\ \vdots & \vdots & \\ 0 & (\varphi_2/\varphi_1)^{(n-1)} & \end{bmatrix}.$$

Expanding along the first column,

$$= \varphi_1^{k+1}\det\left[(\varphi_2/\varphi_1)' \ \ldots \ (\varphi_{k+1}/\varphi_1)'\right].$$

Using the induction hypothesis,

$$\text{ord}_z(\varphi_1^{k+1} \cdot \det\left[(\varphi_2/\varphi_1)' \ \ldots \ (\varphi_{k+1}/\varphi_1)'\right])$$

$$= \text{ord}_z(\varphi_1^{k+1}) + \text{ord}_z(\det\left[(\varphi_2/\varphi_1)' \ \ldots \ (\varphi_{k+1}/\varphi_1)'\right])$$

$$= (k+1)\mu_1 + \sum_{j=2}^{k+1} ((\text{ord}_z(\varphi_j/\varphi_1)') - (j-1) + 1).$$

But, $\text{ord}_z(\varphi_j/\varphi_1)' = \text{ord}_z(\varphi_j/\varphi_1) - 1$

$$= \mu_j - \mu_1 - 1.$$

So,

$$= (k+1)\mu_1 + \sum_{j=2}^{k+1} ((\mu_j - \mu_1 - 1) - (j-2))$$

$$= (k+1)\mu_1 + \sum_{j=2}^{k+1} (\mu_j - \mu_1 + 1 - j)$$

$$= \mu_1 + \sum_{j=2}^{k+1} (\mu_j - j + 1) = \gamma(z) \text{ as desired. } \square$$

From this theorem, there are two corollaries which become apparent.

orollary 2: Let A be a finite-dimensional space of holomorphic functions on domain $D \subset \mathbb{C}$. The set of $z \in D$ with positive weight with respect to A is iscrete.

roof: By this, we mean that around each point with positive weight in D we an find an open set U such that no other such points lie in U.

If z is a point with positive weight, then from the proposition, $\mathrm{rd}_z W_g = \gamma(z) > 0$. We know that $W_g(z)$ is a holomorphic function, and since it s non-zero in this case, we can use the fact that if f is holomorphic on $\subset \mathbb{C}$, $f \neq 0$, then the zeros of f are discrete. Thus, the zeros of $W_g(z)$ are iscrete. □

orollary 3: Under the hypothesis of Corollary 2, for an open dense set in D, he basis $\{\varphi_1, \ldots, \varphi_n\}$ of A adapted to z has the property $\mathrm{ord}_z \varphi_j = j - 1$.

roof: By hypothesis, $\mathrm{ord}_z \varphi_j = \mu_j$. Because $W_g(z)$ is holomorphic with a iscrete set of zeros, on an open dense set the order of vanishing of $W_g(z)$ is . Thus, $\mathrm{ord}_z W_g = 0$ which implies from Proposition 10 that $\gamma(z) = 0$, i.e. $\sum_{=1}^{n} (\mu_j - j + 1) = 0$. But, since $\mu_j = \mathrm{ord}_z \varphi_j \geq 0$, the only way for this to ccur is if $\mu_j = j - 1$ for all j. □

Having established these preliminary remarks, we are now in the position o redefine Weierstrass points in terms of weights.

efinition 7: Let S be a compact Riemann surface of genus $g \geq 1$ and let $^1(S)$ be the space of holomorphic differentials on S. A point $P \in S$ is called Weierstrass point if its weight with respect to $\mathcal{H}^1(S)$ is positive.

In other words, knowing that $\mathcal{H}^1(S)$ has dimension g, we can find the nique basis of $\mathcal{H}^1(S)$ adapted to $P \in S$, $\{\varphi_1, \ldots, \varphi_g\}$, such that rd$_P\varphi_1 < \text{ord}_P\varphi_2 < \ldots < \text{ord}_P\varphi_g$. Let $\mu_j = \text{ord}_P\varphi_j$. Then, P is a Weierstrass oint if $\tau(P) = \sum_{j=1}^{g} (\mu_j - j + 1) > 0$.

At the moment, it may seem that this definition may not be compatible ith the one given previously. However,

roposition 11: A point P on a Riemann surface S of genus $g \geq 2$ is a eierstrass point (by this new definition) if and only if $i[gP] > 0$, i.e. if nd only if there exists a non-constant holomorphic differential on S with a ero of order $\geq g$ at P.

roof: Because surfaces of genus 0 and 1 have no Weierstrass points (as we howed previously), we need only assume that $g \geq 2$.

Suppose first that $i[gP] > 0$ so that P is a Weierstrass point under the ld definition. From Theorem 2, we saw that $i[gP] = \dim(L((\omega) - gP) =$ im$\{\omega \mid \omega$ is a d.f.k. such that ω has a zero of order $\geq g$ at P$\}$. Recall that $((\omega) - gP)$ is a subspace of $\mathcal{H}^1(S)$. We know that $\dim(\mathcal{H}^1(S)) = g$. This mplies that a basis for $\mathcal{H}^1(S)$ adapted to P will consist of g linearly ndependent differentials of the first kind, $\{\varphi_1, \ldots, \varphi_g\}$. Moreover, if e let $\text{ord}_P(\varphi_j) = \mu_j$, then $\mu_1 < \mu_2 < \ldots < \mu_g$ by construction of this asis. Now, consider $\tau(P) = \sum_{j=1}^{g} (\mu_j - j + 1)$. Because $i[gP] > 0$, this implies that there is a differential of the first kind π on S with a zero of order $\geq g$ at P. This differential will be a finite linear combination of $\{\varphi_1, \ldots, \varphi_g\}$. Thus, in order for $ord_P\pi \geq g$, at least one of the μ_j for some j must be $\geq g$. (Although this is not true for an arbitrary basis of $\Omega^1(S)$, because we have constructed an adapted basis, this fact follows.) Hence, for that j, $\mu_j - j + 1 \geq g - j + 1 > 0$ since j can only be one of $1,\ldots,g$. Now, since we know that $\mu_k \geq k - 1$ for all k, this implies that $\gamma(P) > 0$, and P is a Weierstrass point under the most recent definition.

Suppose P is a Weierstrass point under the new definition, so $\gamma(P) > 0$. This implies that for some j, $\mu_j > j - 1 \geq 1$. But, because our basis is arranged such that $\mu_1 < \mu_2 < \ldots < \mu_g$ and since $\mu_k \geq k - 1$ for all k, the fact that at j, $\mu_j \geq j$ implies that $\mu_{j+1} \geq j + 1$, which implies $\mu_{j+2} \geq j + 2$. Continuing in this vein, for any integer k, $\mu_{j+k} \geq j + k$. Now, if $j + k = g$, then $\mu_g \geq g$. Thus, $ord_P\varphi_g \geq g$ which implies that $\varphi_g \in L((\omega) - gP)$ and $i[gP] = \dim(L((\omega) - gP) > 0$. Therefore, P is a Weierstrass point under the old definition. □

Because of this theorem, we now know that indeed the two definitions given above for Weierstrass points are compatible. The fact that we have two different manners in which to view the same concept at our disposal will prove to be quite advantageous for our future endeavors. Complicated theoretical manipulations under one definition may be significantly simplified when approached from the other direction. Such is the case when attempting to determine the estimates on the number of Weierstrass points, W, on a compact Riemann surface S. When this was mentioned previously, proof was not attempted at that point because under the first definition the result is not quite obvious. Approaching this problem from the standpoint of weights, owever, the desired result becomes almost an immediate consequence. herefore, using our second definition of Weierstrass points, we will stablish the following:

heorem 13: Let W be the number of Weierstrass points on a compact Riemann urface of genus $g \geq 2$. Then, $2g + 2 \leq W \leq g^3 - g$.

Before proving this, there are two preliminary results which must be stablished.

roposition 12: For $g \geq 2$, let $\gamma(P)$ be the weight of $P \in S$ with respect to $^1(S)$. Let W_g be the Wronskian of a basis for $\mathcal{H}^1(S)$, whose dimension is $\text{im}(\mathcal{H}^1(S)) = g$. Then, W_g is a holomorphic $g(g+1)/2$ - differential. Hence, $\sum_{\in S} \gamma(P) = (g - 1)g(g + 1)$.

roof: First, let us show that W_g is indeed a $g(g+1)/2$ - differential, i.e. hat W_g is of the form $f(z)(dz)^{g(g+1)/2}$.

Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $\mathcal{H}^1(S)$, such that $\omega_i = f_i(z)dz$ in ocal coordinates, where f_i is holomorphic. Then, by definition,

$$W_g = \det \begin{bmatrix} \omega_1 & & \omega_g \\ f_1'(dz)^2 & \cdots\cdots & f_g'(dz)^2 \\ f_1''(dz)^3 & \vdots & f_g''(dz)^3 \\ f_1^{(g-1)}(dz)^g & & f_g^{(g-1)}(dz)^g \end{bmatrix}$$

here $f_i^{(k)}(dz)^{k+1}$ is a $(k + 1)$ - differential. Then, W_g is a sum of terms f the form $f(dz)(dz)^2 \ldots (dz)^g$ where f is some holomorphic function. ut, $f(dz)(dz)^2 \ldots (dz)^g = f(dz)^{1+2+\ldots+g} = f(dz)^{g(g+1)/2}$. Therefore, $_g$ is a $g(g+1)/2$ - differential.

Now, it can be checked that W_g transforms as a $g(g+1)/2$ - differential.

For an m-differential, the number of zeros counting multiplicities is (2g - 2). Since W_g is a $g(g+1)/2$ - differential, $\sum_{P\in S} \mathrm{ord}_P W_g =$ g-1)2(g(g+1)/2) = (g-1)g(g+1). But, since $\sum_{P\in S} \mathrm{ord}_P W_g = \sum_{P\in S} \gamma(P)$, this implies $\sum_{\in S} \gamma(P) = (g-1)g(g+1)$ as desired. □

heorem 14: For $g \geq 2$, the weight of a point with respect to the holomorphic belian differentials is $\leq g(g-1)/2$. This bound is attained only for a point where the "non-gap" sequence begins with 2.

roof: Proposition 12 tells us that (1) $\sum_{P\in S} \gamma(P) = (g-1)g(g+1)$. This gives s a bound on the number of Weierstrass points on S. The larger the weights t $P \in S$, the smaller will be the number, W, of Weierstrass points.

Now, (1) implies that $\gamma(P) \leq (g-1)g(g+1)$. However, we can determine an ven more precise estimate for $\gamma(P)$. Let $2 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_g = 2g$ be he first g "non-gaps" at P. Then, $1 = n_1 < n_2 < \ldots < n_g < 2g$ are the g gaps" at P. (This sequence $\{n_j\}$ is just the complement of $\{\alpha_1, \ldots, \alpha_g\}$ n $\{1, \ldots, 2g\}$.) Then, recalling that the possible orders of abelian ifferentials of the first kind at P are:

$$0 = n_1 - 1 < n_2 - 1 < \ldots < n_g - 1 \leq 2g - 2,$$

his implies that in a basis $\{\omega_1, \ldots, \omega_g\}$ for $\mathcal{H}^1(S)$, $\mathrm{ord}_P \omega_j = \mu_j = n_j - 1$, hich gives, $n_j = \mu_j + 1$. Hence,

$$\gamma(P) = \sum_{j=1}^{g} (\mu_j - j + 1) = \sum_{j=1}^{g} (n_j - j) = \sum_{j=1}^{g} n_j - \sum_{j=1}^{g} j.$$

et, $\sum_{j=1}^{g} (\alpha_j + n_j) = \sum_{j=1}^{2g} j$, which implies $\sum_{j=1}^{g} n_j = \sum_{j=1}^{2g} j - \sum_{j=1}^{g} \alpha_j$. Thus,

$$\gamma(P) = \sum_{j=1}^{2g} j - \sum_{j=1}^{g} \alpha_j - \sum_{j=1}^{g} j = \sum_{j=g+1}^{2g} j - \sum_{j=1}^{g} \alpha_j = \sum_{j=g+1}^{2g-1} j - \sum_{j=1}^{g-1} \alpha_j .$$

Now, $\sum_{j=1}^{2g-1} j = \sum_{j=1}^{g} j + \sum_{j=g+1}^{2g} j$. From this, we get

$$2g(2g - 1)/2 = g(g - 1)/2 + \sum_{j=g+1}^{2g} j .$$

So,

$$\sum_{j=g+1}^{2g-1} j = 2g^2 - g + (g^2 - g)/2 = 3/2\, g^2 - 3/2\, g = 3/2\, g(g - 1).$$

However, $\sum_{j=g+1}^{2g-1} j \leq \sum_{j=g+1}^{2g} j = 3/2\, g(g - 1)$. Also, from Corollary 1, $\sum_{j=1}^{g-1} \alpha_j \geq g(g - 1)$, which implies $-\sum_{j=1}^{g-1} \alpha_j \leq -g(g - 1)$. Therefore,

$$\tau(P) = \sum_{j=g+1}^{2g-1} j - \sum_{j=1}^{g-1} \alpha_j \leq 3/2\, g(g - 1) - g(g - 1) = g(g - 1)/2.$$

Now, if $\alpha_1 = 2$, $\sum_{j=1}^{g-1} \alpha_j = g(g - 1)$, and we get equality. □

At this point, we are ready to prove Theorem 13.

Proof of Theorem 13:

We know from proposition 12 that $\sum \gamma(P) = (g - 1)g(g + 1)$. This gives us an upper bound on W. Thus, $W \leq g^3 - g$.

Now, from Theorem 14, $\gamma(P) \leq g(g - 1)/2$. Also, the larger the weights at $P \in S$, the fewer the number of Weierstrass points on S. Suppose that for all P on S, $\gamma(P) = g(g - 1)/2$. Then, if W is the number of Weierstrass points, W will be the smallest possible on S in this case. Also, we have: $\sum_{P \in S} \gamma(P) = W(g(g - 1)/2) = g(g - 1)(g + 1)$. So, $W = 2g + 2$. Therefore, this is a lower bound on W, so $2g + 2 \leq W$. □

We now have a rough estimate for W. Interestingly enough, we can describe situations wherein equality holds in the expression of Theorem 13. Recall from previously that for a hyperelliptic curve C, the "gap" sequence or a Weierstrass point on that curve is $\{1,3,5, \ldots, 2g-1\}$. But, this mplies that the "non-gap" sequence is $\{2,4, \ldots, 2g\}$. Since $\alpha_1 = 2$ is he first "non-gap" listed here, by Theorem 14, $\gamma(P) = g(g-1)/2$ for all $\in S$. Hence, as we saw in the proof of Theorem 13, this implies $W = 2g + 2$.

Suppose that the "gap" sequence at each Weierstrass point P of a surface is $1,2, \ldots, g-1, g+1$. Then, as we found in proving Thoerem 14,

$(P) = \sum_{j=1}^{g} (n_j - j)$ where n_j is a "gap". Then, for

$n_j) = \{1,2, \ldots, g-1, g+1\}$, $\gamma(P) = \sum_{j=1}^{g-1} 0 + ((g+1) - g) = 1$.

herefore, $\sum_{P \in S} \gamma(P) = W(\gamma(P)) = g(g-1)(g+1)$ which implies that $= g^3 - g$.

[. Background Information: Singular Curves

A. Differentials and the Riemann-Roch Question

Having discussed the theory of differentials and Weierstrass points on nooth algebraic curves, it would be appropriate at this time to move to a imilar discussion on singular (especially nodal) curves. Our general eference for singular curves is [11]. Suppose we let X be an irreducible ingular algebraic curve in $\mathbb{P}^2$, or $\mathbb{P}^r$, $r \geq 2$ more generally. Then, we know ne following:

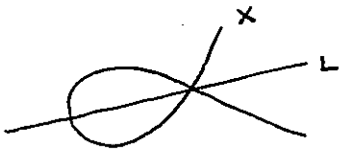
__act__: There exists a smooth algebraic curve (or Riemann surface) $\overline{X}$, called ne __normalization__ of X, with the properties that:

1) The field of rational functions on X is isomorphic to $K(\overline{X})$; and

2) There exists a "parametrization" of X by $\overline{X}$: that is, there is a olomorphic mapping $\pi: \overline{X} \longrightarrow X$ such that for some finite set of points $_1, \ldots, P_n$ of X, $\pi: \overline{X} - \pi^{-1}(\{P_1, \ldots, P_n\}) \longrightarrow X - \{P_1, \ldots, P_n\}$ is n isomorphism.

__xamples__:

1) If we consider the nodal cubic X defined by $y^2 = x^3 + x^2$ in $\mathbb{P}^2$, this as normalization $\overline{X} = \mathbb{P}^1$. To see this, suppose we let $L: y = tx$ be any line hrough the node P of X. Then, L meets X with multiplicity 2 at P. To find he third point of intersection, solve $y^2 = x^3 + x^2$ and $y = tx$ imultaneously.



$$(tx)^2 = x^3 + x^2$$

$$t^2 = x + 1$$

$$x = t^2 - 1$$

$$y = t(t^2 - 1)$$

ıis defines a holomorphic mapping $\bar{\pi}: \mathbb{P}^1 \longrightarrow X$, $\bar{\pi}(t) = (t^2 - 1, t^3 - t)$, ıere $\pi: \mathbb{P}^1 - \pi^{-1}\{P\} \longrightarrow X - \{P\}$ is an isomorphism. Thus, $\mathbb{P}^1$ is the ırmalization of X.

2) Similarly, in considering the cuspidal cubic $X: y^2 = x^3$, this also has ırmalization $\overline{X} = \mathbb{P}^1$. Again, letting $L: y = tx$ be any line (isomorphic to ·) through the cusp Q, solving the equations for X and L simultaneously, we .nd: $(xt)^2 = x^3$, which implies $x = t^2$ and $y = t^3$. Thus, $\pi: \mathbb{P}^1 \longrightarrow X$ defined ı $\pi(t) = (t^2, t^3)$ is a holomorphic mapping and $\pi: \mathbb{P}^1 - \pi^{-1}\{Q\} \longrightarrow X - \{Q\}$ is ı isomorphism as desired.

3) The curve $X: y^2 = x^2(x + 3)(x + 2)(x + 1)$, which graphically is as in ıe figure, has as normalization a curve of genus 1. An explicit holomorphic ıpping $\pi: \overline{X} \longrightarrow X$ would involve elliptic functions.

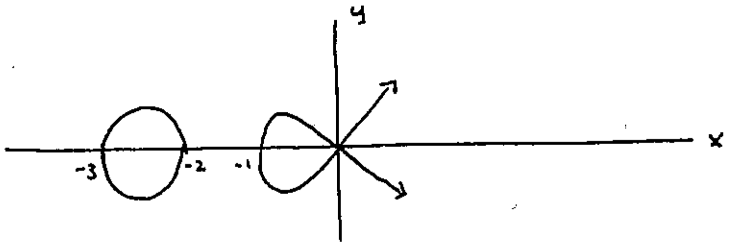


Fig. 6

4) For any g, we can construct a <u>g-nodal rational nodal curve</u> X by ıoosing 2g distinct points b_i and c_i in $\mathbb{P}^1$ and identifying them in pairs. A ·nodal rational nodal curve would be:

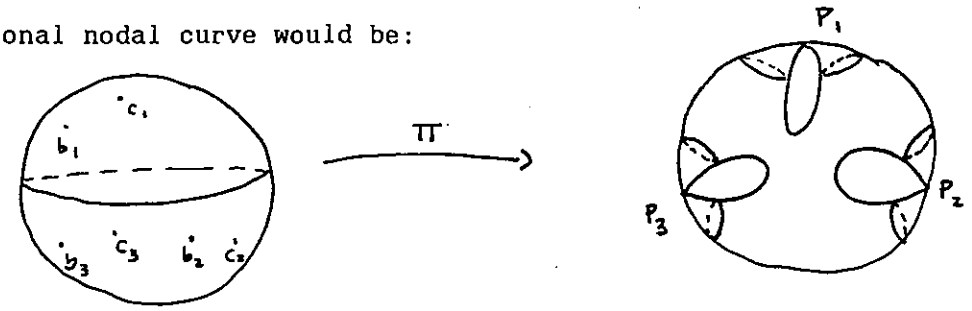


Fig. 7

uch a curve has normalization $\mathbb{P}^1$.

Just as in the case of smooth algebraic curves, we also have the notion f the genus of a singular curve.

efinition 8: If X is a singular curve, and $\overline{X}$ is its normalization, with arametrization $\pi: \overline{X} \longrightarrow X$, then the geometric genus of X is the genus of $\overline{X}$ s a smooth algebraic curve.

It would be advantageous to us if X actually behaved like a smooth lgebraic curve of genus equal to the geometric genus. We have previously iscussed much of the theoretical background concerning these smooth algebraic urves. However, this is not the case. In fact, singular curves actually ave properties similar to smooth curves of genus bigger than the geometric enus of X. Moreover, the more "complicated" the singular points, the larger his "arithmetic genus" of the singular curve tends to be. Before we give a recise definition of "arithmetic genus", it is necessary to provide a means f measuring the degree of complexity of the singularity of X at P, called the -invariant. For our purposes, it is more beneficial to define this value for he particular types of singularities which we will encounter in our ensuing liscussion.

efinition 9: 1) The δ-invariant for a smooth point P is $\delta_P = 0$.

2) For a node (ordinary double point), $\delta_P = 1$.

3) If P is a "unibranch singularity" (If U is a small open set f the curve such that $P \in U$, then P is a unibranch singularity if

$^{1}(U - \{P\})$ consists of one connected component in $\overline{X}$.), we can define δ_P as
llows: Let $\mathcal{O}_P$ be the local ring of X at P (Recall, a local ring is a mmutative ring with identity that has exactly one maximal ideal.),
$= \{f \in K(X) \mid f$ has no pole at $P\}$, and let $\overline{\mathcal{O}}$ be the corresponding ring at $^{1}(P) \in \overline{X}$, i.e. $\overline{\mathcal{O}} = \{g \in K(\overline{X}) \mid g$ has no pole at $\pi^{-1}(P)\}$. Then, we define $= \dim_{\mathbb{C}}(\overline{\mathcal{O}}/\mathcal{O}_P)$. (This quotient is always a finite-dimensional vector space 'er $\mathbb{C}$.)

amples: 1) Consider the example of the cusp X: $y^2 = x^3$, given previously. this case, $\overline{X} = \mathbb{P}^1$ and $\pi(t) = (t^2,t^3)$. In this case, for the singularity $= 0$ on X, corresponding to $t = 0$ on $\mathbb{P}^1$, $\overline{\mathcal{O}} = \{a_0+a_1t+a_2t^2+ \ldots \mid a_i \in \mathbb{C}\}$. the other hand, because X has parametrization $\pi: \mathbb{P}^1 \longrightarrow X$ defined by $(1,t) = (1,t^2,t^3)$, we see that $\mathcal{O}_P = \{b_0+b_2t^2+b_3t^3+ \ldots \mid b_i \in \mathbb{C}\}$. Thus, $\mathrm{im}(\overline{\mathcal{O}}/\mathcal{O}_P) = 1$, i.e. $\overline{\mathcal{O}}/\mathcal{O}_P$ is generated by t. Therefore, $\delta_P = 1$ in this ase.

2) In general, for a cusp X: $y^2 = x^{2k+1}$, we find that

$$\overline{\mathcal{O}} = \{a_0+a_1t+a_2t^2+a_3t^3+ \ldots \mid a_i \in \mathbb{C}\} \text{ and}$$

$$\mathcal{O}_P = \{b_0+ b_2t^2+b_4t^4+b_6t^6+\ldots+b_{2k}t^{2k}+b_{2k+1}t^{2k+1}+ \ldots \mid b_i \in \mathbb{C}\}.$$

hus, $\dim(\overline{\mathcal{O}}/\mathcal{O}_P) = k$ and $\delta_P = k$.

With the notion of the δ-invariant fresh in mind, we are now in the osition to define the "arithmetic genus" of X.

efinition 10: The arithmetic genus of X is given by:

$$p_a(X) = g(\overline{X}) + \sum_{P\in X} \delta_P .$$

Notice that $\sum_{P\in X} \delta_P$ does indeed converge. If $P \in X$ is a smooth point,

= 0. Moreover, on a singular curve, there are only a finite number of ıgularities, with finite values for δ_P. Hence, $\sum_{P\in X} \delta_P$ is actually a finite ı of integers.

amples:

1) The nodal cubic $X: y^2 = x^3 + x^2$ was seen to have normalization $\mathbb{P}^1$. us, $g(\mathbb{P}^1) = 0$. Moreover, the node P is its only singularity, with $\delta_P = 1$. nce, $p_a(X) = 1$.

2) For the cuspidal cubic $X: y^2 = x^3$, we saw that $\overline{X} = \mathbb{P}^1$, so again $\overline{X})=0$. Also, we found that at the cusp Q, $\delta_Q = 1$. So, $p_a(X) = 1$.

3) Considering $X: y^2 = x^{2k+1}$, which also has normalization $\mathbb{P}^1$, we saw at $\delta_P = k$ at the cusp P. Hence, $p_a(X) = k$.

4) Suppose we look at $X: y^2 = x^2(x+3)(x+2)(x+1)$, the "nodal hyperelliptic ırve", which has a node at $P = (0,0)$ and no other singularities. In this ıse, $g(\overline{X}) = 1$ and $\delta_P = 1$, so $p_a(X) = 1 + 1 = 2$.

5) Finally, for a g-nodal rational curve X, $g(\overline{X}) = 0$. Also, X has g ɔdes $P_1, \ldots, P_g$ at the points where b_i and c_i are identified, with $\delta_i = 1$ for $i = 1,\ldots,g$. These nodes are the only singularities of X. ence, $p_a(X) = 0 + \sum_{i=1}^{g} \delta_{P_i} = 1 + \ldots + 1 = g$.

Recall that on a smooth algebraic curve, C (compact Riemann surface), of enus g, the vector space $\mathcal{H}^1(C)$ of holomorphic differentials has dimension . We may ask if a similar phenomena occurs on a singular curve, X. In fact, n such an X, the arithmetic genus also measures the dimension of a vector pace of differentials, called dualizing differentials.

Definition 11: Let X be an irreducible singular algebraic curve in $\mathbb{P}^r$, $r \geq 2$, and let $\overline{X}$ be its normalization. A dualizing differential on X is a meromorphic differential on $\overline{X}$, ω, with the following properties:

1) ω has no poles except at $Q \in \pi^{-1}$({singular points of X});

2) for all singular points $P \in X$, $\sum_{Q\in\pi^{-1}(P)} \operatorname{Res}_Q(f\omega) = 0$ for all $f \in \mathcal{O}_P$.

It is interesting to note here that as for a meromorphic differential ω on a smooth algebraic curve C of degree g for which $\deg((\omega)) = 2g-2$, so too do we have a similar formula for a dualizing differential ω of a singular curve of arithmetic genus $p_a(X)$: $\deg((\omega)) = 2p_a(X) - 2$ where (ω) is the divisor of ω.

Examples:

1) As an example, suppose we consider the nodal cubic $X: y^2 = x^3 + x^2$, which can be viewed as $\overline{X} = \mathbb{P}^1$ with the points P=1, Q=-1 identified. Let π be the parametrization mentioned in the previous example concerning this curve. It can be shown that the differential of the third kind

$\omega_{+1,-1} = dz/(z-1) + (-dz)/(z+1)$ on $\mathbb{P}^1$ gives a dualizing differential on X. For X, the singularity is P = 0, with $\pi^{-1}(P) = \{1,-1\}$. Notice that $\omega_{+1,-1}$ has poles only at +1 and -1, the elements of $\pi^{-1}(P)$. Recalling that

$\operatorname{Res}_{+1}(\omega_{+1,-1}) = 1$ and $\operatorname{Res}_{-1}(\omega_{+1,-1}) = -1$, and given any $f \in \mathcal{O}_0$, we have:

$$\sum_{Q\in\pi^{-1}(P)} \operatorname{Res}_Q(f\omega_{+1,-1}) = \operatorname{Res}_{+1}(f\omega_{+1,-1}) + \operatorname{Res}_{-1}(f\omega_{+1,-1}) = f(0) - f(0) = 0$$

as desired. Since $f \in \mathcal{O}_P$ was arbitrary, $\omega_{+1,-1}$ is a dualizing differential.

2) Similarly, if X is a g-nodal rational curve formed by identifying b_i

and c_i on $\mathbb{P}^1$, $i=1,\ldots,g$ for distinct points b_i, c_i, the differentials ω_{b_i,c_i} all give dualizing differentials on X.

3) If $g(\overline{X}) > 0$, then every differential ω of the first kind on $\overline{X}$ also gives a dualizing differential on X. Such differentials ω have no poles, and for all $f \in \mathcal{O}_P$, f has no pole at P. Hence, $\operatorname{Res}_Q(f\omega) = 0$ for all $Q \in \pi^{-1}(P)$. (Notice that $g(\overline{X}) > 0$. This is because if $g(\overline{X}) = 0$, then $\dim(\mathcal{H}^1(\overline{X})) = 0$ and there are no differentials of the first kind on $\overline{X}$.)

4) On the cuspidal curve $X: y^2 = x^3$, the differential of the second kind on $\mathbb{P}^1$, $\omega = dt/t^2$, is a dualizing differential. The only singularity of X is at $P = (0,0)$, corresponding to $t = 0$ on $\overline{X}$. Notice that ω has a pole at $t = 0$ and no other poles. Moreover, recalling that $\mathcal{O}_0 = \{a_o+a_2t^2+a_3t^3+\ldots \mid a_i \in \mathbb{C}\}$, and noticing that $\operatorname{Res}_{t=0}(t^k(dt/t^2)) = 0$ for $k = 0,2,3,\ldots$ we see that for any $f \in \mathcal{O}_P$, $\operatorname{Res}_{t=0}(f\omega) = 0$. Hence, ω is a dualizing differential on X.

5) For $X: y^2 = x^{2k+1}$, it can be seen that the following will be a set of dualizing differentials for X: $L = \{dt/t^2, dt/t^4, dt/t^6, \ldots, dt/t^{2k}\}$. Notice that at the sole singularity $P = (0,0)$ of X, corresponding to $t = 0$ in $\mathbb{P}^1$, each of these differentials $\omega_i = dt/t^{2i}$ has a pole, but no other poles. Recall that $\mathcal{O}_P = \{a_o+a_2t^2+a_4t^4+a_6t^6+\ldots+a_{2k}t^{2k}+a_{2k+1}t^{2k+1}+\ldots \mid a_i \in \mathbb{C}\}$. Then, $\operatorname{Res}_{t=0}(t^j(dt/t^{2i})) = 0$ for all $j = 2,4,6,\ldots,2k,2k+1,\ldots$ and all $i = 1,\ldots,k$. Hence, for any $f \in \mathcal{O}_P$ and all $i = 1,\ldots,k$, $\operatorname{Res}_{t=0}(f\omega_i) = 0$ and L is a set of dualizing differentials of X.

In general, we have:

<u>Theorem 15</u>: The dualizing differentials on X form a vector space of dimension $p_a(X)$ over $\mathbb{C}$.

Thus, in the five examples given above, we have actually found all of the dualizing differentials in each case.

Having established this parallel between concepts on a smooth algebraic curve and those on a singular curve, we now venture to pose the Riemann-Roch question on singular curves. The answer to the Riemann-Roch question is very nice in the case in which the singularities of X are "Gorenstein". For instance, nodal curves are Gorenstein, as are curves possessing only cusps locally isomorphic to $y^2 = x^3$. And so, we have:

<u>Theorem 16</u>: (Riemann-Roch Theorem for Gorenstein Curves)

Let D be a divisor on X such that D contains no singular points of X. Then,

$$\dim(L(D)) - \dim(L((\omega) - D) = \deg(D) + 1 - p_a(X)$$

where (ω) denotes the divisor of a dualizing differential on X.

B. Weierstrass Points and the Abel Question

Recall that if Y is a smooth algebraic curve over $\mathbb{C}$ (Riemann surface) of genus $g \geq 2$, and K is the canonical divisor class on Y (i.e. the divisor class of the divisors of differentials of the first kind on Y), then a point $P \in Y$ is a ("classical") Weierstrass point if there exists a differential of the first kind, ω, on Y such that $(\omega) - gP \geq 0$, so that $\dim(L(K - gP) > 0$. In a similar fashion, if $n > 1$, let $\gamma_n = \dim(L(nK)) = (2n-1)(g-1)$ by the Riemann-Roch formula. Then, we say that $P \in Y$ is a Weierstrass point of order n (or an n-Weierstrass point) if there is an n-differential η such that $(\eta) - \gamma_n P \geq 0$, so $\dim(L(nK - \gamma_n P) > 0$. In other words, η has a zero of order γ_n at P, which is a zero of a higher order than expected. (We know that there exist n-differentials η with zeros of order $< \gamma_n = \dim(L(nK))$.) (From our earlier discussion, we know that these definitions may also be stated equivalently in terms of the Wronskian of a basis of the vector space of differentials of the first kind, $\mathcal{H}^1(S)$, or the vector space of n-differentials; or in terms of weights.)

Above, K was assumed to be the canonical divisor class on Y. More generally, we can consider the situation in which D is <u>any</u> divisor on Y. Then, P is a <u>Weierstrass point of order n of D</u> if either:

1) $\dim(L(nD - s_n P)) > 0$, where $s_n = \dim(L(nD))$; or
2) there exists an $f \in L(nD)$ with a zero of order at least s_n at P.

If we consider a divisor D on a smooth algebraic curve Y such that $\deg(D) > 0$, and let W(D) represent the set of all Weierstrass points of D of order n for all n, i.e. $W(D) = \bigcup_{n=1}^{\infty} \{P \mid P \text{ is a W-point or order } n \text{ of } D\}$, then it is known, by a result of B. Olsen ([10]), that W(D) is dense in Y (in its usual topology). As we have done before, we may ask if notions such as these for smooth curves can be extended to similar concepts in the case of a singular curve. In fact, such a development is indeed possible. We begin by investigating the idea of Weierstrass points on a singular curve.

Originating with the work of R.F. Lax and C. Widland, [14], we have a manner in which to define Weierstrass points on Gorenstein curves. Let X be an irreducible projective Gorenstein curve of arithmetic genus $g > 0$ over $\mathbb{C}$, and let ω be the canonical divisor class of X (i.e., the divisor class of divisors of dualizing differentials). Suppose D is a divisor $\Sigma n_i P_i$ on X such that for all i, P_i is not one of the singular points on X, and let $s = \dim(L(D))$ defined as on a smooth curve. Then, we can find a basis $\{\varphi_1, \ldots, \varphi_s\}$ for L(D). Fix a function $\varphi^{(\alpha)}$ whose poles on the coordinate chart U_α of X are no worse than $D|_{U_\alpha}$ = the points of D in U_α. Expanding in local coordinates on $U_\alpha \subset X$, for each j we have:

$$\varphi_j = F_{1,j}^{(\alpha)} \varphi^{(\alpha)} = \left(\sum_{k=0}^{\infty} a_{j,k}^{(\alpha)} z^k \right) \varphi^{(\alpha)}$$

where z is the local coordinate on U_α and $F_{1,j}^{(\alpha)}$ is a function possessing no poles in U_α. Then, $d\varphi_j$ are dualizing differentials, and we have:

$$\begin{aligned} d\varphi_j = dF_{1,j}^{(\alpha)} & \text{ which we define to be } F_{2,j}^{(\alpha)}\, dz \\ dF_{2,j}^{(\alpha)} & \text{ defined as } F_{3,j}^{(\alpha)}\, dz \\ & \vdots \\ dF_{i-1,j}^{(\alpha)} & \text{ which we let equal } F_{i,j}^{(\alpha)}\, dz \end{aligned}$$

From this, we can construct the Wronskian:

$$\rho^{(\alpha)} = \det \begin{bmatrix} F^{(\alpha)}_{1,1} & \cdots & F^{(\alpha)}_{1,s} \\ F^{(\alpha)}_{2,1} & \cdots & F^{(\alpha)}_{2,s} \\ \vdots & \ddots & \vdots \\ F^{(\alpha)}_{s,1} & \cdots & F^{(\alpha)}_{s,s} \end{bmatrix} (\varphi^{(\alpha)})^{s}(dz)^{s(s-1)/2}$$

which, as in the case of a smooth curve is an expression which is invariant under coordinate changes. In other words, $\rho^{(\alpha)} = \rho^{(\beta)}$ on $U_\alpha \cap U_\beta$. Hence, we can define ρ to be an element of the space of s(s-1)/2 - differentials on X with local expansion $\rho^{(\alpha)}$ on U_α such that the poles of ρ are bounded by sD.

With this construction of ρ, we are in the position to be able to discuss the Weierstrass points of X. First, we need:

Definition 12: The order of vanishing of ρ at $P \in X$ is equal to the order of vanishing of the Wronskian $\det(F^{(\alpha)}_{1,j})$ at P if $P \in U_\alpha$.

(Recall that the order of vanishing of a differential f(z)dz is given by the order of vanishing of f. Furthermore, the order of vanishing of f at a point P, $\mathrm{ord}_P(f)$, is the integer k such that the coefficient a_ℓ of X in the Laurent expansion of f around P corresponding to x = 0 is zero for $\ell < k$, but $a_k \neq 0$.)

We now have all of the machinery to state the following definition:

Definition 13: 1) The D-Weierstrass weight of P is the order of vanishing of ρ at P. (This value is usually 0.)

2) P is a <u>Weierstrass point</u> (of D) if the D-Weierstrass eight of P is > 0.

3) P is a <u>Weierstrass point of order n</u> of D if P is a W-point f nD.

Quite expectedly, as on smooth curves, it can be shown that a smooth oint P on X is a W-point of D if and only if $\dim(L(D - sP) > 0$ where $s =$ im(L(D)).

From the definitions given above, it is interesting to note that a recise value for the "number" of W-points of D (where the "number" is counted ith weight as its multiplicity) can be found.

<u>roposition 13</u>: ([7], Prop. 2, p. 109) The "number" of W-points of D, ounting multiplicities, is

$$s(\deg(D)) + s(s-1)(g-1).$$

i.e., Σ weights $= s(\deg(D)) + s(s-1)(g-1)$.)

<u>roof</u>: Notice that $\rho/(dz)^{s(s-1)/2}$ is an element of $L((s(s-1)/2)\cdot(\omega) + sD)$. ence, $\rho = f\eta$ where $f \in L(sD)$ and η is an $s(s-1)/2$ - differential. We ould like to determine the order of vanishing of ρ, i.e. the number of zeros hat ρ has on X. This value will be the order of vanishing of all Wronskians, nd hence the total weight.

Since $f \in L(sD)$, f has at most $s(\deg(D))$ poles, and because f is a eromorphic function on X, this implies that f also has at most $s(\deg(D))$ eros. On the other hand, the fact that ω is an $s(s-1)/2$ - differential ndicates that ω has $s(s-1)/2\ (\deg((\omega))) = s(s-1)(2g-2)/2 = s(s-1)(g-1)$ zeros. ince $\rho = f\omega$, the number of zeros of ρ is equal to the sum of the number of eros of f and of ω, i.e. the number of zeros of $\rho = s(\deg(D)) + s(s-1)(g-1)$.

ence, the total weight is s(deg(D)) + s(s-1)(g-1) as desired. □

Defining W(D) again to be the set of all Weierstrass points of order n or all n, and recalling the result of Olsen that on a smooth curve on which he divisor D had positive degree W(D) is dense, the question can be raised as what the behavior of W(D) would be on a singular curve X. On such curves, large portion of the total "number" of Weierstrass points is accounted for y the singular points of X. In fact, we have:

roposition 14: ([7], Prop. 3, p. 110) Let P be a singular point with -invariant δ_P. Then the Weierstrass weight of P is $\geq s(s-1)\delta_P$.

orollary 4: If s > 1, then every singular point is a Weierstrass point of D.

roof: Since P is a singular point, $\delta_P > 0$. Hence, by the proposition, the eierstrass weight of P is $\geq s(s-1)\delta_P > 1(0)(0) = 0$, and P is a Weierstrass oint by definition. □

From this, it may be conjectured that as a family of smooth algebraic urves approaches a singular curve, many of the Weierstrass points of these mooth curves tend toward the singularities. This seems to suggest that on a ingular curve, the set W(D) might no longer be dense. Throughout the emainder of this paper, we will be working with 2-nodal rational nodal urves, attempting to justify this claim and in fact determining the exact ocation of the limit points of the set W(D). Before we begin this task, owever, we must discuss a few more preliminaries.

Beginning in the general case, let X be an irreducible rational nodal curve with $g > 1$ nodes. We seek an answer to the Abel question on X: Given a divisor D of degree 0 on X which does not contain any of the singularities of X, when is D the divisor of a meromorphic function on X? The answer is surprisingly simple in this case.

Since X is a rational nodal curve, we have seen that $\overline{X} = \mathbb{P}^1$ is the normalization of X, with the parametrization:

$$\pi: \overline{X} \longrightarrow X = \overline{X}/c_i{\simeq}b_i,\ i=1,..,g\ .$$

Notice that a meromorphic function on X is merely a meromorphic function on $\mathbb{P}^1$ which takes the same value at b_i as at c_i for each $i = 1,\ldots,g$. On the other hand, if D is a divisor of degree 0 on X which does not contain any of the nodes of X, we can view D as a divisor on $\mathbb{P}^1$. Hence, there exists a function $f \in K(\mathbb{P}^1)$ such that $(f) = D$ on $\mathbb{P}^1$. (Recall that on $\mathbb{P}^1$, $\deg(D) = 0$ is the only condition for the existence of such a function.) Moreover, this f will be an element of K(X) if and only if $f(b_i) = f(c_i)$ for all i. We can restate this discussion equivalently in terms of mappings.

Suppose we define the mapping:

$$\begin{array}{lll} i: \mathrm{Div}^0(X) & \longrightarrow & (\mathbb{C}^*)^g =: J, \text{ the Jacobian of } X \\ D & \longrightarrow & (f(b_1)/f(c_1),\ \ldots\ ,f(b_g)/f(c_g)) \end{array}$$

where $\mathrm{Div}^0(X) = \{D$ of degree 0 on X, not containing any nodes$\}$ and $f \in K(\mathbb{P}^1)$ is such that $(f) = D$. Then, it is easily shown that i is a group homomorphism from the additive group $\mathrm{Div}^0(X)$ to the multiplicative group $(\mathbb{C}^*)^g$.

<u>Theorem 17</u>: (Abel's Theorem for X) $D \in \mathrm{Div}^0(X)$ is the divisor of a function in $K(X)$ if and only if $D \in \mathrm{Ker}(i)$, if and only if $i(D) = (1,\ldots,1) \in (\mathbb{C}^*)^g$.

There is yet another manner in which to express this result. If we let $D = \Sigma n_k x_k$ be an effective divisor, we can construct a divisor of degree 0 by choosing any smooth point $x_o \in X$ and forming $D - \deg(D)x_o$. Then, we have a second mapping: $\Psi\colon \mathrm{Div}(X) \longrightarrow (\mathbb{C}^*)^g =: J$ where $\mathrm{Div}(X) = \{$all D on X, D contains no nodes$\}$ defined by $\Psi(D) = i(D - \deg(D)x_o)$. In particular, suppose D is an effective divisor of degree 1, i.e., $D = x \in X$, and take $x_o = \infty$. Consider the function $f(t) = t - x$. Notice that $(f) = x - \infty = D - \infty$. Then, by definition,

$$\begin{aligned}\Psi(D) = i(D - \infty) &= (f(b_1)/f(c_1),\ldots,f(b_g)/f(c_g)) \\ &= ((b_1-x)/(c_1-x),\ldots,(b_g-x)/(c_g-x)).\end{aligned}$$

Because i is a group homomorphism, we can extend Ψ to a map on all effective divisors:

$$\Psi(\Sigma n_k x_k) = \left(\prod_k ((b_1-x_k)/(c_1-x_k))^{n_k},\ldots,\prod_k ((b_g-x_k)/(c_g-x_k))^{n_k}\right)$$

This mapping Ψ is called the <u>Abel-Jacobi mapping</u>.

For those more familiar with the theory of smooth algebraic curves, the name of this mapping should ring a bell. In fact, on a smooth algebraic curve S of genus 2, the Abel mapping $\gamma\colon S \longrightarrow \mathbb{C}^2/\Lambda$ is defined by $\gamma(P) = \left(\int_{P_o}^{P}\omega_1, \int_{P_o}^{P}\omega_2\right)(\mathrm{mod}\,\Lambda)$ where P_o is some fixed point of S, and ω_1 and ω_2 are the basis elements of the vector space $\mathcal{H}^1(S)$ of all differentials of the first kind on S. It may be asked if this mapping, when applied to X, is the ame as that defined by $\tilde{\varphi}$. In fact, we can show that the mapping on X_o = smooth points of X} $\varphi: X_o \longrightarrow (\mathbb{C}^*)^2$ defined by (x) = $((x-b_1)/(x-c_1),(x-b_2)/(x-c_2))$ presented by Lax for singular curves X f genus 2 is in fact equivalent to the Abel mapping γ above when γ is applied o X, a singular curve of arithmetic genus 2.

For a 2-nodal rational nodal curve formed by identifying b_1 and c_1, b_2 nd c_2 on $\mathbb{P}^1$, we have seen that the dualizing differentials are:

$_1 = dx/(x-b_1) - dx/(x-c_1) = (b_1-c_1)dx/(x-b_1)(x-c_1)$ and

$_2 = dx/(x-b_2) - dx/(x-c_2) = (b_2-c_2)dx/(x-b_1)(x-c_2)$. Moreover, ω_i has poles t b_i and c_i and no other poles, such that $\operatorname{Res}_{b_i} \omega_i = +1$, and $\operatorname{Res}_{c_i} \omega_i = -1$.

First, we must establish that the spaces $(\mathbb{C}^*)^2$ and $\mathbb{C}^2/\Lambda$, where Λ is he lattice of periods of ω_1 and ω_2, are isomorphic as groups, and describe he isomorphism. Recall that in the case of a smooth algebraic curve of genus , there are four cycles A_1,A_2,B_1,B_2 around which the basis vectors $\{\omega_1,\omega_2\}$ f Ω_S are integrated to construct the lattice of periods of ω_1 and ω_2.

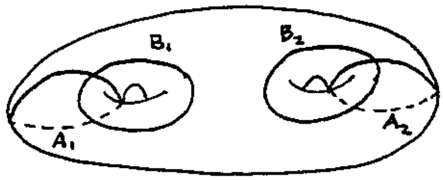


Fig. 8

owever, notice that when we construct a 2-nodal rational curve, two of these eriods are lost in the identification process on $\mathbb{P}^1$, namely the periods a_1 nd a_2 in the preceeding diagram are deformed to a single point. Hence, in orming Λ for X, there are but two cycles to integrate ω_1 and ω_2 around. In act, more can be said about the generators of Λ. Let P_o be any fixed base oint in X, such that $P_o \neq b_1,c_1,b_2,c_2$. Consider: A_1 and A_2 are the two ycles around which ω_1 and ω_2 are to be integrated. Since $\overline{X} = \mathbb{P}^1$, and we have

Fig. 9

parametrization $\pi: \mathbb{P}^1 \longrightarrow X$, suppose we pull A_1 and A_2 back to $\mathbb{P}^1$. Then, $\pi^{-1}(A_i)$ either winds once around b_i or once around c_i in $\mathbb{P}^1$. Let's suppose $\pi^{-1}(A_i)$ winds around b_i for $i = 1,2$. (The other cases are the same.) Then,

$$\int_{\pi^{-1}(A_1)} \omega_1 = 2\pi i(\operatorname{Res}_{b_1}(\omega_1)) = 2\pi i; \qquad \int_{\pi^{-1}(A_2)} \omega_1 = 0$$

$$\int_{\pi^{-1}(A_2)} \omega_2 = 2\pi i(\operatorname{Res}_{b_2}(\omega_2)) = 2\pi i; \qquad \int_{\pi^{-1}(A_1)} \omega_2 = 0.$$

lence, the period lattice is generated by $(2\pi i,0)$ and $(0,2\pi i)$. Thus,

$\Lambda = \langle(0,2\pi i),(2\pi i,0)\rangle$, and $\mathbb{C}^2/\Lambda = \mathbb{C}^2/\langle(0,2\pi i),(2\pi i,0)\rangle$

$$\simeq \mathbb{C}/\langle(0,2\pi i)\rangle \oplus \mathbb{C}/\langle(2\pi i,0)\rangle.$$ But,

onsider the following sequence:

$$0 \xrightarrow[i_1]{\text{inclusion}} \langle 2\pi i\rangle \xrightarrow[i_2]{\text{inclusion}} \mathbb{C} \xrightarrow{e^z} \mathbb{C}^* \xrightarrow[i_3]{} \{1\}.$$

his is an exact sequence. Therefore, we know (from the theory of exact equences) that $\mathbb{C}^* \simeq \mathbb{C}/\langle 2\pi i\rangle$. Hence, $\mathbb{C}^2/\Lambda \simeq (\mathbb{C}^*)^2$ as desired.

With this in mind, we would now like to show that γ and φ actually map ny point $P \in X_0$ to the same element of $(\mathbb{C}^*)^2$. Let $\Lambda = \langle(2\pi i,0),(0,2\pi i)\rangle$ and onsider the sequence:

$$\begin{array}{ccccccccc} 0 & \xrightarrow[i_1]{\text{inclusion}} & \Lambda & \xrightarrow[i_2]{\text{inclusion}} & \mathbb{C}^2 & \xrightarrow{\quad} & (\mathbb{C}^*)^2 & \xrightarrow[i_4]{\quad} & \{1\} \\ & & & & (z,w) & \xrightarrow{i_3} & (e^z, e^w) & & \end{array}$$

ıis is an exact sequence. Hence, $\mathbb{C}^2/\Lambda \simeq (\mathbb{C}^*)^2$ under the isomorphism i_3. ıw, suppose $P \in X_o$ is any point in X_o. Then,

$$\begin{array}{ccccc} {}_o & \longrightarrow & \mathbb{C}^2/\Lambda & \longrightarrow & (\mathbb{C}^*)^2 \\ \mapsto & & \left(\int_{p_o}^{p} \omega_1, \int_{p_o}^{p} \omega_2 \right) = (\log((x-b_1)/(x-c_1)), \log((x-b_2)/(x-c_2)) & \rightarrow & (e^{\int \omega_1}, e^{\int \omega_2}) \end{array}$$

ıerefore, $\gamma(P) = ((x-b_1)/(x-c_1), (x-b_2)/(x-c_2)) = \Upsilon(P)$ as desired and the ɔel mapping and the map defined by Lax actually coincide for 2-nodal rational ɔdal curves. □

With this proof, we see that in fact the Abel mapping for smooth lgebraic curves has an equivalent formulation on rational nodal curves X. In ldition to this observation about Υ, there are some other facts to notice:

<u>ıcts</u>: 1) On effective divisors of degree 1, Υ gives an injection

: $X - \{\text{nodes}\} \longrightarrow (\mathbb{C}^*)^g$.

2) If $1 < m < g$, Υ gives a mapping:

$$\Upsilon: \text{EffDiv}^m(X) \longrightarrow (\mathbb{C}^*)^g$$

ıere $\text{EffDiv}^m(X)$ is the set of all effective divisors of degree m on X, such ıat the image of Υ is an m-dimensional subset of $(\mathbb{C}^*)^g$.

Because $\Upsilon(\text{EffDiv}^{g-1}(X))$ has dimension g-1 in $(\mathbb{C}^*)^g$, we expect that the ıage space will be the zero set of one analytic function of g variables. In act, we can explicitly describe this function:

$$\tau_X(\lambda_1,\ldots,\lambda_g) = \det\begin{bmatrix} 1-\lambda_1 & \cdots & 1-\lambda_g \\ b_1-c_1\lambda_1 & \cdots & b_g-c_g\lambda_g \\ \vdots & & \vdots \\ b_1^{g-1}-c_1^{g-1}\lambda_1 & & b_g^{g-1}-c_g^{g-1}\lambda_g \end{bmatrix}$$

et Θ denote the zero set of τ_X. $(\Theta \subset (\mathbb{C}^*)^g)$.

emma 1: Suppose $x_1,\ldots,x_g \in X_o$. Then $x_1 + \ldots + x_g - \infty$ is (equivalent to) n effective divisor (of degree g-1) if and only if

$$\tau_X(\varphi(x_1 + \ldots + x_g - \infty)) = 0.$$

roof: I refer the reader to [9, p.3.251]. □

orollary 5: $\tau_X(\varphi(\mathrm{EffDiv}^{g-1}(X))) = 0$.

roof: Suppose $x_1 + \ldots + x_{g-1} \in \mathrm{EffDiv}^{g-1}(X)$. Then, $_1 + \ldots + x_{g-1} + \infty - \infty$ is equivalent to an effective divisor and by lemma 1, $_X(\varphi(x_1 + \ldots + x_{g-1})) = 0$. □

With this mapping τ_X, we now can state a result fundamental to the study f the distribution of the Weierstrass points on X. In fact, this lemma gives criteria for establishing when a point $P \in X$ is a (smooth) Weierstrass point f a given divisor D of X. Namely:

<u>emma 2</u>: ([7], Lemma 2, p. 113) Suppose $g = 2$. Suppose D is a divisor on X uch that $L((\omega) - D) = \{0\}$, i.e. D is not special. Let $s = \dim(L(nD)) > 0$. P s a smooth Weierstrass point of D of order n if and only if $\varphi(D)(\varphi(P))^{-s} =$ $\lambda_1, \lambda_2)$ satisfies $\tau_x(\lambda_1, \lambda_2) = 0$.

<u>roof</u>:

Since D is not special, by the Riemann-Roch formula we have: $= \dim(L(D)) = \deg(D) + 1 - g$. We can rewrite this as:

a) $$\deg(D) - s = g - 1.$$

When P is a Weierstrass point of D, $\dim(L(D - sP)) > 0$. Therefore, we an find an $f \in L(D - sP)$. Consider the effective divisor $D' = (f) + D - sP$. Notice this is effective since (f) includes a factor $s'P$ where $s' \geq s$, ince $f \in L(D - sP)$. Hence, this factor $s'P$ cancels the factor sP of D', aking the coefficients of all other points P of D' positive.) We have that $eg(D') = \deg((f)) + \deg(D) - \deg(sP) = \deg(D) - s$ since $\deg((f)) = 0$ ecause $f \in L(D - sP)$. When we are in the situation in which $g = 2$, by (a) bove we have: $\deg(D') = \deg(D) - s = g - 1 = 2 - 1 = 1$. Thus, $D' = Q \in X$, nd by Corollary 5, $\varphi(D') \in \Theta$. But,

$$\begin{aligned} \varphi(D') &= \varphi((f) + D - sP) \\ &= \varphi((f))\varphi(D)\varphi(-sP) \text{ since } \varphi \text{ is a group homomorphism} \\ &= (1,\ldots,1)\varphi(D)\varphi(-sP) \text{ by the Abel Theorem} \\ &= \varphi(D)(\varphi(P))^{-s}. \end{aligned}$$

ence, $\varphi(D)(\varphi(P))^{-s} \in \Theta$ if P is a Weierstrass point as desired. □

Earlier in this section, it was hypothesized that the set

$$(D) = \bigcup_{n=1}^{\infty} \{P \mid P \text{ is a W-point of } D \text{ of order } n \text{ on } X\}$$ was not dense on X. In

act, R.F. Lax established the validity of this claim for a specific case of a

-nodal rational nodal curve in [7]. Considering the rational nodal curve X ormed by identifying the points -3 and 0, -1 and 1 on $\mathbb{P}^1$, Lax found that here exists a divisor D on X such that none of the Weierstrass points of D on lie within the disk $|z - 5/6| = 1/6$ on $\mathbb{P}^1$. Hence, W(D) is not dense in . The question still remains, however, where the Weierstrass points of X are ctually located. It is with this question which we will concern ourselves in he next section.

II. Distribution of Weierstrass Points on 2-nodal Rational Curves

A. A Specific Example

Consider the 2-nodal rational nodal curve X formed by identifying the oints $b_1 = -3$ and $c_1 = 0$, $b_2 = -1$ and $c_2 = 1$ on $\mathbb{P}^1$. On this curve, it is nown that the set W(D) of all Weierstrass points of all orders n is not ense. Using numerical methods, it is possible to locate the Weierstrass oints of X and plot their distribution in $\mathbb{P}^1$. The result which allows us to o this is Lemma 2, stated in the previous section.

In the example under consideration, we are in the situation in which $= 2$, and by the lemma, the smooth Weierstrass points of a divisor D are the oints P such that $\Psi(D)(\Psi(P))^{-s} = (\lambda_1, \lambda_2)$ satisfies $\tau_X(\lambda_1, \lambda_2) = 0$. To egin our investigation, we must determine D, $\Psi(D)(\Psi(P))^{-s}$, and $\tau_X(\lambda_1, \lambda_2)$. By efinition,

$$\tau_X(\lambda_1, \lambda_2) = \det \begin{bmatrix} 1-\lambda_1 & 1-\lambda_2 \\ -3 & -1-\lambda_2 \end{bmatrix} = 2 + \lambda_1 - 4\lambda_2 + \lambda_1\lambda_2.$$

onsider $f(z) = (z + 3)z(z + 1)(z - 1) + k$, where $k \in \mathbb{C}$ is nonzero. This is rational function on $\mathbb{P}^1$ such that $f(-3) = f(0) = f(-1) = f(1) = k$. Let D be he divisor of zeros of f(z) on X. Then, $\deg(D) = 4 > 2g - 2 = 2$. which mplies $\dim(L((\omega) - D) = 0$, as is required by the lemma. Because D is not of egree 0, we must consider $\Psi(D - 4\cdot\infty) = \Psi(D)(\Psi(\infty))^{-4}$. But, $(\infty) = ((b_1 - \infty)/(c_1 - \infty), (b_2 - \infty)/(c_2 - \infty)) = (1,1)$. Moreover, since $\mathrm{eg}(D) = 4 = \deg(4\cdot\infty)$, this implies $\deg(D - 4\cdot\infty) = 0$ and for some $f \in K(X)$, $- 4\cdot\infty = (f)$. Therefore, $D \equiv 4\cdot\infty$ and by the Abel Theorem, $\Psi(D) = \Psi(4\cdot\infty) =$ $1,1)$. Thus, $\Psi(D)(\Psi(P))^{-s} = (1,1)(\Psi(P))^{-s} = (\Psi(P))^{-s}$. Moreover, $(P) = ((b_1 - P)/(c_1 - P), (b_2 - P)/(c_2 - P)) = ((P + 3)/P, (P + 1)/(P - 1))$.

herefore, $(\tilde{\mathcal{F}}(P))^{-s} = ((P/(P+3))^s, ((P-1)/(P+1))^s)$. From the lemma, the mooth W-points of X of order n are those $z \in \mathbb{P}^1$ satisfying:

1) $$0 = 2 + (z/(z+3))^s - 4((z-1)/(z+1))^s + (z/(z+3))^s((z-1)/(z+1))^s,$$

here $s = \dim(L(nD)) = 4n-1$ by the Riemann-Roch formula for singular curves.

After writing a program using Newton's method to determine the zeros of his function for an arbitrary n, it is possible to, and we did, generate a raphical representation of the n-W-points of X for n = 1,...,8. (See table nd figure.)

In considering the first few values of n, a pattern seems to emerge. As increases, the W-points of order n seem to tend toward the lines $\mathrm{Re}(z) = 0$ nd $\mathrm{Re}(z) = -3/2$. Since W(D) is an infinite set of points on a compact urface, we know that W(D) has a set of limit points. The above onsiderations suggest the following:

<u>heorem 8</u>: The smooth limit points of the set W(D) for the D constructed bove on the rational nodal curve X formed by identifying the points -3 and 0, 1 and 1 in $\mathbb{P}^1$, lie on the perpendicular bisectors $\mathrm{Re}(z) = -3/2$ and $\mathrm{Re}(z) = 0$ f the segments $\overline{-3,0}$ and $\overline{-1,1}$ respectively.

<u>roof</u>: The proof of this theorem will consist of two parts, each of which ill establish that within a certain region, none of the limit points of W(D) an be found.

1) We first consider the polynomial $f_n(z)$ obtained by clearing nominators in (1):

$$(z) = z^s(z - 1)^s + z^s(z + 1)^s - 4(z - 1)^s(z + 3)^s + 2(z + 1)^s(z + 3)^s.$$

e roots of this polynomial are the smooth W-points of X of order n in the nite part of $\mathbb{P}^1$. (There is also a Weierstrass point of weight 2 at ∞ for l n.) We claim that none of these roots lie within the strip

/2 < Re(z) < 0.

For any z within the strip, $|z + 1| < |z - 1|$. Using the triangle equalities,

$$|f_n(z)| \geq 4|z - 1|^s|z + 3|^s - |z|^s|z - 1|^s - |z|^s|z + 1|^s - 2|z + 1|^s|z + 3|^s.$$

w, for z in the strip, the inequality given above implies

$|f_n(z)| > 2|z - 1|^s(|z + 3|^s - |z|^s)$. But, in the strip, $|z + 3| > |z|$. nce, $|z + 3|^s - |z|^s > 0$ for all s. Therefore, $|f_n(z)| > 0$ for all integers which indicates that $f_n(z) \neq 0$ for all z in the strip.

2) To complete the proof of the theorem, we will show that there exist rcles to the right of Re(z) = 0 (and to the left of Re(z) = -3/2) in which e no W-points of X for n large enough and that as $s = 4n-1 \longrightarrow \infty$, these rcles approach the desired vertical line. Then, in conjunction with part , this will imply that the limit points of W(D) lie on Re(z) = 0 and Re(z) = /2.

The function $g_n(z)$, the zeros of which are n-W-points of X, can be ctored as:

$$) \; g_n(z) = (z/(z+3))^s((z-1)/(z+1))^s + (z/(z+3))^s - 4((z-1)/(z+1))^s + 2$$
$$= (z/(z+3))^s((z-1)/(z+1))^s + (z/(z+3))^s - 4((z-1)/(z+1))^s - 4 + 6$$

$$= ((z-1)/(z+1))^s[(z/(z+3))^s - 4] + [(z/(z+3))^s - 4] + 6$$

$$= [((z - 1)/(z + 1))^s + 1][(z/(z + 3))^s - 4] + 6$$

e want to find regions in which lie no zeros of this, given s.

Suppose we rewrite (2), taking absolute values of both sides:

3) $$|((z - 1)/(z + 1))^s + 1||(z/(z + 3))^s - 4| = 6.$$

f the product on the left of (3) is < 6 (or > 6) for all z in some region, hen there will be no zeros in that region.

Let $u = (z - 1)/(z + 1)$. By the triangle inequality, we have:

$$|((z - 1)/(z + 1))^s + 1| = |u^s + 1| \le |u|^s + 1.$$

or small $|u| < 1$, we can make $|u^s + 1|$ arbitrarily close to 1. But, if $= (z - 1)/(z + 1)$, then $z = (u + 1)/(1 - u)$. Hence, $z/(z + 3)$ can be xpressed as $(u + 1)/(4 - 2u)$ and $|(z/(z + 3))^s - 4|$ becomes $((u + 1)/(4 - 2u))^s - 4|$ in the u-plane. However, as $|u|$ gets smaller, $(u + 1)/(4 - 2u)| \longrightarrow 1/4$. Thus, given any s, $|((u + 1)/(4 - 2u))^s|$ is ounded above, i.e. there exists a bound B(s) such that if $|u| \le B(s)$, then $((z - 1)/(z + 1))^s + 1||(z/(z + 3))^s - 4| < 6$ and there are no zeros inside $u| = r \le B(s)$.

Above, we saw that $|u^s + 1| \le |u|^s + 1 \le (B(s))^s + 1$. Notice that as $\longrightarrow \infty$, we can take B(s) closer to one, and the desired inequality will still old. As $s \longrightarrow \infty$, we must consider $|u| = r \longrightarrow 1^-$ in the u-plane.

Notice that $z(u) = (u + 1)/(1 - u)$ is a linear fractional ransformation. Suppose we consider the domain $|u| < 1/2$. Under z(u), the nterior of $|u| = 1/2$ maps to the interior of $|z - 5/3| = 4/3$ in the -plane. (See Fig. 10.)

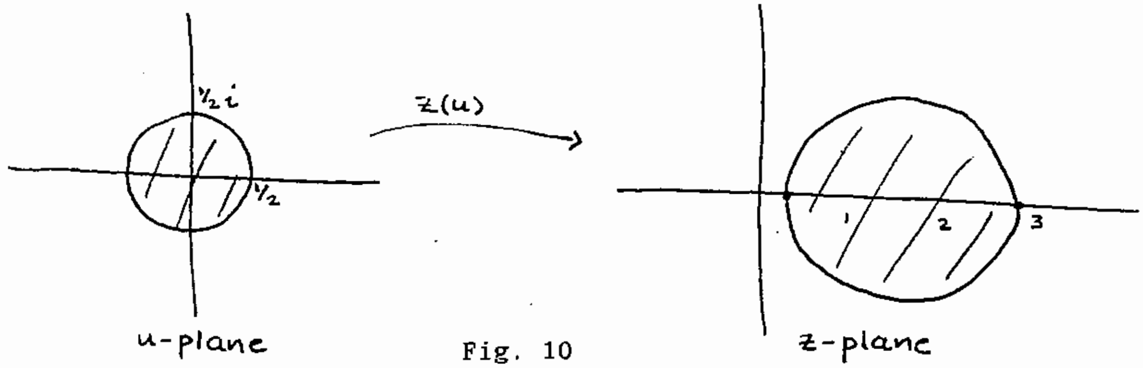


Fig. 10

n the other hand, the image of the interior of $|u| = 1$ under $z(u)$ is the half lane $\operatorname{Re}(z) > 0$. (See Fig. 11.)

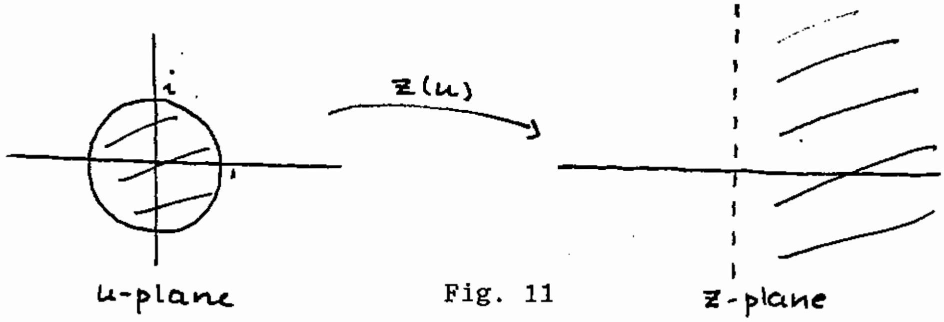


Fig. 11

Thus, as $|u| \longrightarrow 1^-$, the circles approach the vertical line $\operatorname{Re}(z) = 0$. otice that as $s \longrightarrow \infty$, $B(s) \longrightarrow 1^-$. Hence, as $s \longrightarrow \infty$, the circles approach e(z) = 0. Because none of the W-points for n sufficiently large lie within hese circles, none of the limit points of W(D) lie to the right of $\operatorname{Re}(z) = 0$ s $n \longrightarrow \infty$, and with part 1), this implies they lie on $\operatorname{Re}(z) = 0$.

On the other hand, suppose we let $v = z/(z + 3)$. By the triangle nequality, $|(z/(z + 3))^s - 4| = |v^s - 4| \geq |v|^s - 4$. So, for large values f $|v|$, $|v^s| - 4$ can be made arbitrarily large. Again, if $v = z/(z + 3)$, then $= 3v/(1 - v)$. Then, $(z - 1)/(z + 1)$ can be written as $(4v - 1)/(2v + 1)$. ence, $|((z - 1)/(z + 1))^s + 1|$ becomes $|((4v - 1)/(2v + 1))^s + 1|$ in the -plane. Notice that as $|v|$ gets larger, $|(4v - 1)/(2v + 1)| \longrightarrow 2$. Thus, iven any s, $|((4v - 1)/(2v + 1))^s + 1|$ is bounded below, i.e. there exists a ound B(s) such that if $|v| \geq B(s)$, then $((z-1)/(z+1))^s + 1||(z/(z+3))^s - 4| > 6$ and none of the zeros lie outside of he circles $|v| = k \geq B(s)$.

Notice that since $|v^s - 4| \geq |v|^s - 4 \geq (B(s))^s - 4$, as $s \longrightarrow \infty$, we can take $B(s)$ closer to one to ensure that $|((4v-1)/(2v+1))^s + 1||v^s - 4| > 6$ still holds.

Suppose, as before, we consider the images of some circles in the v-plane as transformed under $z(v) = 3v/(1 - v)$ to the z-plane. First, look at $|v| = 2$.

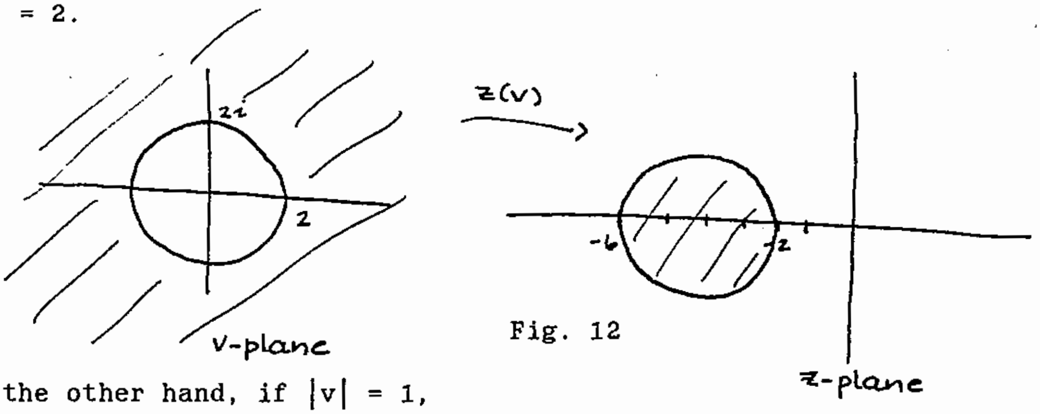


Fig. 12

On the other hand, if $|v| = 1$,

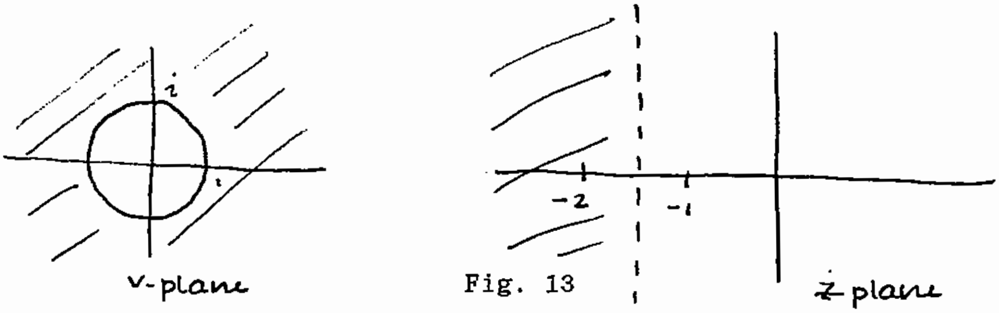


Fig. 13

Thus, as $|v| \longrightarrow 1^+$, i.e. as $s \longrightarrow \infty$, the images of the circles in the v-plane under $z(v)$ approach the vertical line $Re(z) = -3/2$. Moreover, none of the limit points lie to the left of $Re(z) = -3/2$, since none of the n-W-points lie outside of the circles $|v| = k \leq B(s)$ for all s and the exterior of $|v| = 1$, corresponding to s at ∞, maps to the left of $Re(z) = -3/2$. From the results of part 1), this indicates that the limit points of W(D) lie on the line $Re(z) = -3/2$, as well as on $Re(z) = 0$. □

What is the significance of the fact that the limit points of W(D) lie on the vertical lines $Re(z) = 0$ and $Re(z) = -3/2$? Recall that in the case of a ooth curve (Riemann surface), every point on the surface was a limit point f a sequence of W-points. In other words, W(D) was dense on S. However, ıen we move to this particular case of a 2-nodal rational nodal curve on ıich there exist singularities, the situation proves to be quite different. ıre, the set of all n-W-points is not dense on X. Rather, the limit points le strictly on the vertical lines Re(z) = 0 and Re(z) = -3/2.

B. The General Case

Having established the validity of the claim that the limit points of the t of all n-Weierstrass points for the rational nodal curve X formed by entifying $b_1 = -3$ and $c_1 = 0$, $b_2 = -1$ and $c_2 = 1$ on $\mathbb{P}^1$, lie on the rpendicular bisectors of the segments $\overline{b_1c_1}$ and $\overline{b_2c_2}$, we may ask what the ture of the situation is for a general 2-nodal rational curve. Whereas in r specific example the points of identification were all real, in the neral case, b_1, c_1, b_2, and c_2 would be any points on the projective line $\mathbb{P}^1$. on identification of these points, a rational nodal curve of genus 2 will ve been constructed on which there exist singularities, each of which is a ierstrass point of high weight for the curve. However, as in our specific se, there also exist n-Weierstrass points which are not singularities. ing an infinite set and lying on a compact surface, the set W(D) of all ierstrass points of order n for all n has a set of limit points. To locate ese points, we again turn to Lax's discussion. Notice that for our specific se, the tool which allowed us to locate the W-points using numerical methods s a lemma which applied to <u>any</u> rational nodal curve. Hence, we can also use is result (Lemma 2) in our present discussion.

According to the lemma, the smooth Weierstrass points of a divisor D are e points P such that $\Psi(D)(\Psi(P))^{-s} = (\lambda_1, \lambda_2)$ satisfies $\tau_x(\lambda_1, \lambda_2) = 0$. By finition,

$$\begin{aligned}\tau_x(\lambda_1, \lambda_2) &= \det\begin{bmatrix} 1-\lambda_1 & 1-\lambda_2 \\ b_1-c_1\lambda_1 & b_2-c_2\lambda_2 \end{bmatrix} \\ &= (1-\lambda_1)(b_2-c_2\lambda_2) - (1-\lambda_2)(b_1-c_1\lambda_1) \\ &= b_2 - c_2\lambda_2 - b_2\lambda_1 + c_2\lambda_1\lambda_2 - b_1 + c_1\lambda_1 - b_1\lambda_2 - c_1\lambda_1\lambda_2 \\ &= (b_2 - b_1) + (c_1 - b_2)\lambda_1 + (b_1 - c_2)\lambda_2 + (c_2 - c_1)\lambda_1\lambda_2\end{aligned}$$

onsider $f(z) = (z - b_1)(z - c_1)(z - b_2)(z - c_2) + k$ where $k \in \mathbb{C}$ is nonzero. his is a rational function on $\mathbb{P}^1$ such that $f(b_1) = f(c_1) = f(b_2) = f(c_2) = k$. et D be the divisor of zeros of f(z) on X. Then, because $\deg(D) = 4 > 2g-2 =$, $\dim(L((\omega) - D)) = 0$ as desired. Because D is not of degree 0, we must onsider $\varphi(D - 4\cdot\infty) = \varphi(D)\varphi(\infty)^{-4}$. But, we saw in the previous section that $(\infty) = (1,1)$. Moreover, since $\deg(D) = 4 = \deg(4\cdot\infty)$, as before in the example f the previous section, $D \equiv 4\cdot\infty$. Hence, $\varphi(D) = \varphi(4\cdot\infty) = (1,1)$ and $(D)(\varphi(P))^{-s} = (\varphi(P))^{-s}$. By definition, $\varphi(P) = ((b_1-P)/(c_1-P),(b_2-P)/(c_2-P))$. herefore, $(\lambda_1,\lambda_2) = (((c_1-P)/(b_1-P))^s,((c_2-P)/(b_2-P))^s)$, and from Lemma 2, he smooth Weierstrass points of X of order n are those z satisfying:

$$0 = (b_2-b_1) + (c_1-b_2)((c_1-P)/(b_1-P))^s + (b_1-c_2)((c_2-P)/(b_2-P))^s + (c_2-c_1)((c_1-P)/(b_1-P))^s((c_2-P)/(b_2-P))^s$$

here $s = \dim(L(nD)) = 4n-1$ by the Riemann-Roch formula for singular curves.

Using this information, we can prove the following:

<u>**heorem 9**</u>: Assume that b_1,c_1,b_2, and c_2 in $\mathbb{P}^1$ are four distinct points on he projective line. Then, the limit points of the set of all n-Weierstrass oints for the divisor D constructed above of the rational nodal curve X ormed by identifying b_1 and c_1, b_2 and c_2 on $\mathbb{P}^1$, lie on the perpendicular isectors of the segments $\overline{b_1c_1}$ and $\overline{b_2c_2}$.

Before beginning the proof, it is important to realize that there is a ignificant difference between the situation in the case of a smooth algebraic urve and that of the case of a 2-nodal rational curve. Recall from our arlier discussion that for a smooth curve (Riemann surface), every point on he surface was a limit point of a sequence of elements of W(D). In other ords, the set of n-W-points of the surface S for all n was dense on S. owever, we have just seen that there exists a rational nodal curve X and a ivisor D on X such that W(D) is not dense. Yet, this is not a special case. n fact, after numerically analyzing two other situations, each a variation of he first, the hypothesis is given a solid foundation.

In constructing other rational nodal curves for investigation, it was nportant to make each sufficiently different from the first, as well as each ther, in order to better establish that our hypothesis does indeed have ıbstance. First, notice that in the example given previously, the intervals -3,0] and [-1,1] overlap. Hence, as a second example, suppose the points of lentification are: $b_1 = -3$, $c_1 = -1$, $b_2 = 1$, and $c_2 = 2$. In this case, the ıtervals [-3,-1] and [1,2] do not overlap, yet the points are still purely əal. Moreover, using the numerical methods described earlier and graphing ıe results, (see graph and table) we again notice that as n becomes larger, ıe set of n-Weierstrass points approaches the perpendicular bisectors ə(z) = -2 and Re(z) = 3/2 of $\overline{-3,-1}$ and $\overline{1,2}$ respectively. Therefore, it əems that the theorem holds if all of the points of identification are real. ɔwever, we indicated that the points could be any elements of $\mathbb{P}^1$. Realizing ıis, as a third example, the identifications $b_1 = 0$ and $c_1 = 1$, $b_2 = i$ and $_2 = 2$ were made. Upon determining the roots of the appropriate polynomials ıd graphing, we see that the n-Weierstrass points tend to be approaching the ərpendicular bisectors of $\overline{0,1}$ and $\overline{i,2}$ as predicted. It is with this :rong encouragement that we proceed to the proof of the proposed theorem.

<u>:oof of Theorem 9</u>:

We have determined that the equation of the function whose zeros are the

-points of X is given by:

$$f_s(z) = (b_2-b_1) + (b_1-c_2)((c_2-z)/(b_2-z))^s + (c_1-b_2)((c_1-z)/(b_1-z))^s$$
$$+ (c_2-c_1)((c_1-z)/(b_1-z))^s((c_2-z)/(b_2-z))^s$$

here s = 4n-1. Suppose we let $u = (z - c_1)/(z - b_1)$ and let
= $(z - c_2)/(z - b_2)$. Then, we can write $f_s(z)$ as:

1) $$f_s(z) = (b_2-b_1) + (b_1-c_2)v^s + (c_1-b_2)u^2 + (c_2-c_1)u^s v^s$$

llowing s to approach ∞, we would like to find regions in which there lie no -W-points for n sufficiently large.

Suppose we fix z arbitrarily in $\mathbb{P}^1$. This will fix u and v to constants. ow, let $s \longrightarrow \infty$ (i.e., $n \longrightarrow \infty$). There are four cases to consider.

<u>Case 1</u>: Suppose that z is such that $|u| < 1$ and $|v| < 1$. We know that $b_2-b_1|$, $|b_1-c_2|$, $|c_1-b_2|$, and $|c_2-c_1|$ are all > 0, because b_1, b_2, c_1, and $_2$ are all distinct points. Moreover, $|u|^s$ and $|v|^s$ are > 0 for any choice of , but < 1 for all s. Thus, we can find an $s_o = 4n_o - 1$ such that if $s \geq s_o$,

*) $$|c_2 - c_1||u|^s|v|^s < 1/3\ |b_2 - b_1|$$
$$|c_1 - b_2||u|^s < 1/3\ |b_2 - b_1|$$
$$|b_1 - c_2||v|^s < 1/3\ |b_2 - b_1|$$

Notice here that the closer that both $|u|$ and $|v|$ are to 1, the larger ill be the necessary value of s_o to make these inequalities hold. In this ise, to determine s_o, let $A = \max\{|c_2 - c_1|,|c_1 - b_2|,|b_1 - c_2|\}$ and = $\max\{|u|, |v|\}$ (< 1). Then, for some s_o, $AB^{s_o} = 1/3\ |b_2 - b_1|$. Taking le natural logarithm of both sides, we have:

$$\log(AB^{s_o}) = \log(1/3\ |b_2-b_1|)$$

$$\log(A) + s_o\log(B) = \log(1/3) + \log(|b_2-b_1|)$$

$$s_o = (\log(1/3) + \log(|b_2-b_1|) - \log(A))/(\log(B))$$

Keeping (*) in mind and using the triangle inequalities, we have:

$$\begin{aligned} |f_s(z)| &\geq |b_2-b_1| - |(c_2-c_1)u^s v^s + (c_1-b_2)u^s + (b_1-c_2)v^s| \\ &\geq |b_2-b_1| - \{|c_2-c_1||u|^s|v|^s + |(c_1-b_2)u^s + (b_1-c_2)v^s|\} \\ &\geq |b_2-b_1| - |c_2-c_1||u|^s|v|^s - \{|c_1-b_2||u|^s + |b_1-c_2||v|^s\} \\ &= |b_2-b_1| - |c_2-c_1||u|^s|v|^s - |c_1-b_2||u|^s - |b_1-c_2||v|^s \end{aligned}$$

n conjunction with the set of inequalities (*), this implies:

$$|f_s(z)| > |b_2-b_1| - 1/3\ |b_2-b_1| - 1/3\ |b_2-b_1| - 1/3\ |b_2-b_1| = 0.$$

rom this, we find that $f_s(z) \neq 0$. Recall that a point z is an n-W-point of X f and only if $f_s(z) = 0$ where $s = 4n-1$. Thus, considering the above, for ny z such that $|u| < 1$ and $|v| < 1$, there exists an integer $n_o = (s_o+1)/4$ uch that z is <u>not</u> a W-point of order n for all $n \geq n_o$. (If $(s_o+1)/4$ is not n integer, take the next largest integer closest to $(s_o+1)/4$ as the value of $_o$. Then, the results will still hold.) Hence, any z in the region $\{z \mid |u| < 1,\ |v| < 1\}$ is not a limit point of W(D), as it is not even an -W-point of X for n sufficiently large.

Where do such z lie in the plane? If $|u| < 1$ and $|v| < 1$, this implies $|(z-c_1)/(z-b_1)| < 1$ and $|(z-c_2)/(z-b_2)| < 1$, i.e., $|z-c_1| < |z-b_1|$ and $|z-c_2| < |z-b_2|$. Geometrically, we know that the points z such that

$z-b_1| = |z-c_1|$ lie on the perpendicular bisector ℓ_1 of the segment $\overline{b_1c_1}$. ikewise, if z is such that $|z-c_2| = |z-b_2|$, then z lies on the perpendicular isector ℓ_2 of $\overline{b_2c_2}$. Now, since $|z-c_i| < |z-b_i|$ for i = 1,2, such z lie to he right of ℓ_i, i.e. on the side of ℓ_i on which c_i lies. Thus, the situation s as follows:

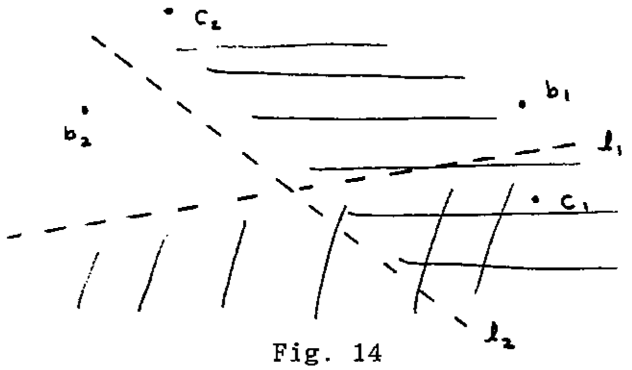


Fig. 14

In the hatched quadrant, none of the limit points of W(D) can be found.

Case 2: Suppose now that z is such that $|u| > 1$ and $|v| > 1$. Before e proceed, since $|u| \neq 0$ and $|v| \neq 0$, which implies $u \neq 0$ and $v \neq 0$, we would ike to rewrite $f_s(z) = 0$ by dividing through by $u^s v^s$:

$$= f_s(z) = (b_2-b_1)(1/u))^s(1/v)^s + (b_1-c_2)(1/u)^s + (c_1-b_2)(1/v)^s + (c_2-c_1).$$

ow, using the triangle inequalities, we have that:

$$\begin{aligned} f_s(z)| &\geq |c_2-c_1| - |(b_2-b_1)(1/u)^s(1/v)^s + (b_1-c_2)(1/u)^s + (c_1-b_2)(1/v)^s| \\ &\geq |c_2-c_1| - \{|b_2-b_1||1/u|^s|1/v|^s + |(b_1-c_2)(1/u)^s + (c_1-b_2)(1/v)^s| \\ &\geq |c_2-c_1| - |b_2-b_1||1/u|^s|1/v|^s - \{|b_1-c_2||1/u|^s + |c_1-b_2||1/v|^s\} \\ &= |c_2-c_1| - |b_2-b_1||1/u|^s|1/v|^s - |b_1-c_2||1/u|^s - |c_1-b_2||1/v|^s. \end{aligned}$$

ecause $|v| > 1$ and $|u| > 1$, we have that $|1/u| < 1$ and $|1/v| < 1$. Therefore, e are back in the situation of case 1. Following a similar argument as bove, we find that for all z in the region $\{z \mid |u| > 1, |v| > 1\}$, z is not limit point of W(D).

In the plane, consider: $|u| > 1$ implies $|(z-c_1)/(z-b_1)| > 1$, i.e. $z-c_1| > |z-b_1|$. Likewise, $|v| > 1$ implies $|(z-c_2)/(z-b_2)| > 1$, i.e. $z-c_2| > |z-b_2|$. From this, we can see that the z which satisfy these onditions lie to the left of both ℓ_1 and ℓ_2, which is the side of ℓ_1 and ℓ_2 n which b_1 and b_2 lie respectively. Pictorially, we have:

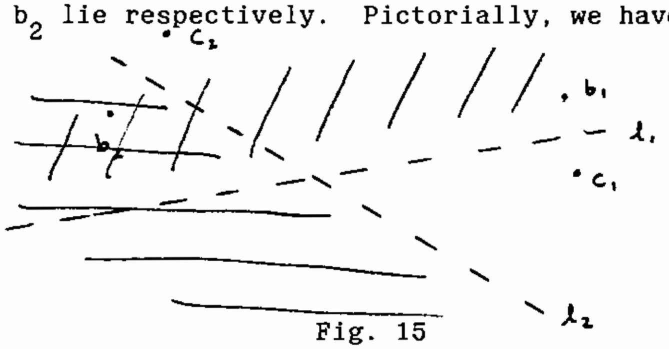


Fig. 15

Within the hatched region, none of the limit points of W(D) can be found.

<u>Case 3</u>: Now, consider the case where z is such that $|u| > 1$ and $|v| < 1$. gain, to bring us back to the familiar case 1, we would like to rewrite (1) y dividing through by u^s, which is allowable since $|u| \neq 0$ implies $u \neq 0$. o, we have:

$$0 = f_s(z) = (c_2-c_1)v^s + (c_1-b_2) + (b_1-c_2)v^s(1/u)^s + (b_2-b_1)(1/u)^s.$$

y the triangle inequalities, we then have:

$$|f_s(z)| \geq |c_1-b_2| - |c_2-c_1||v|^s - |b_1-c_2||v|^s|1/u|^s - |b_2-b_1||1/u|^s.$$

ince $|u| > 1$, we know that $|1/u| < 1$. Also, $|v| < 1$ and we are in the ituation of case 1 again. Under an argument similar to that given for the irst case, we find that none of the limit points of W(D) lie within the egion $\{z \mid |u| > 1, |v| < 1\}$.

Where does this region lie in the z-plane? Since $|u| > 1$, we have $|(z-c_1)/(z-b_1)| > 1$, i.e. $|z-c_1| > |z-b_1|$ and such z lie on the side of ℓ_1 on hich b_1 lies. Also, $|v| < 1$ implies $|z-c_2| < |z-b_2|$. These z lie on the ide of ℓ_2 on which c_2 is found. The picture is thus:

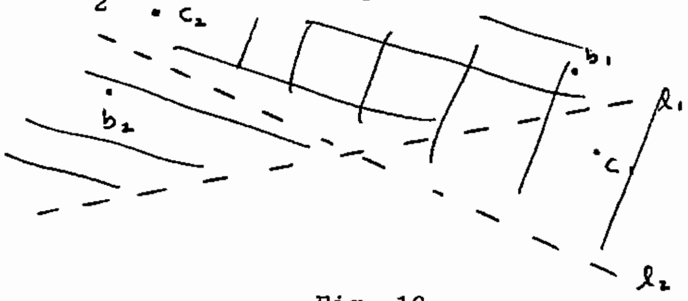


Fig. 16

Again, in the hatched region there do not exist any limit points of W(D).

Case 4: Finally, consider the situation of a z such that $|u| < 1$ and $|v|$ 1. As in cases 2 and 3, we can rewrite $f_s(z)$ by dividing through by $^s(\neq 0)$:

$$0 = f_s(z) = (b_2-b_1)(1/v)^s + (b_1-c_2) + (c_1-b_2)u^s(1/v)^s + (c_2-c_1)u^s.$$

oreover, the triangle inequality yields:

$$|f_s(z)| \geq |b_1-c_2| - |b_2-b_1||1/v|^s - |c_1-b_2||u|^s|1/v|^s - |c_2-c_1||u|^s.$$

ere, since $|v| > 1$, we have that $|1/v| < 1$. In conjunction with the fact hat $|u| < 1$, we again are placed back in the situation of case 1, which mplies that, under similar argumentation, none of the limit points of W(D) ie within the region $\{z \mid |u| < 1, |v| > 1\}$.

To see this pictorially, notice that, as before, $|u| < 1$ implies $|z-c_1| < |z-b_1|$ and such z lie on the side of ℓ_1 on which c_1 lies. Also, $|v| > 1$ implies $|z-c_2| > |z-b_2|$ and these z lie on the side of ℓ_2 on which

b_2 lies. And so, we have:

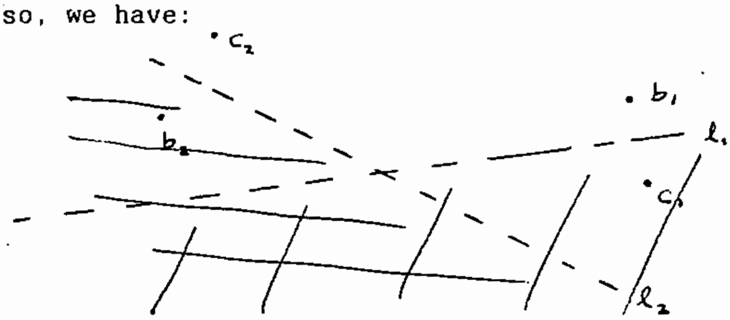


Fig. 17

Within the hatched region, one cannot find any limit points of W(D).

Considering these four cases simultaneously, we see that none of the imit points of the set of all n-W-points of X lie in the four quadrants into hich $\mathbb{P}^1$ is divided by the perpendicular bisectors of $\overline{b_1c_1}$ and $\overline{b_2c_2}$. (See igure). Hence, since there do exist limit points of W(D), these limit points ust lie on ℓ_1 and ℓ_2, the perpendicular bisectors of $\overline{b_1c_1}$ and $\overline{b_2c_2}$ espectively. □

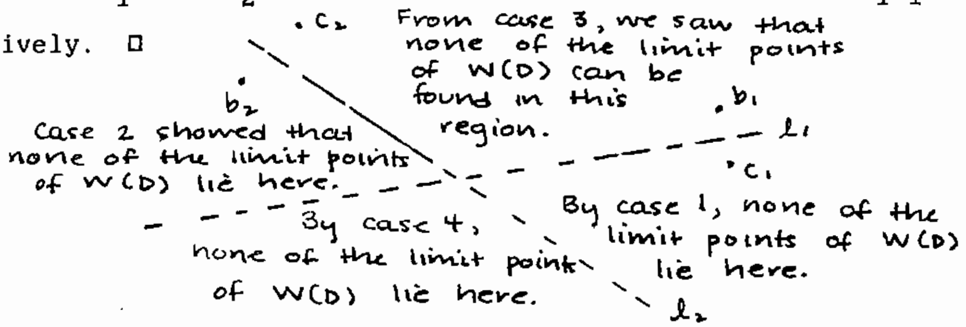


Fig. 18

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Appendix A
Smooth Weierstrass Points for X

n = 1

.1537 ± .5673i

1.9314 ± .7819i

n = 2

.0808 ± .7943i

1.6557 ± .3388i

2.3256 ± 2.9115i

.0520 ± .2277i

.2566 ± 2.0509i

1.7415 ± 1.1783i

n = 3

.0381 ± .4561i

.1075 ± -1.5508i

.0321 ± .1436i

1.8160 ± 2.3047i

.0551 ± .8650i

1.5963 ± .2148i

1.6140 ± .6818i

.3866 ± 3.3560i

1.6647 ± 1.2907i

2.8172 ± 4.8306i

n = 4

.0308 ± .5769i

.0668 ± 1.3743i

.0234 ± .1050i

1.5766 ± .4862i

1.5923 ± .8636i

1.5700 ± .1573i

1.6999 ± 2.0517i

.0418 ± .8995i

.0255 ± .3247i

1.6252 ± 1.3454i

.1392 ± 2.2388i

-1.9039 ± 3.3229i

.5280 ± 4.6271i

-3.3321 ± 6.7220i

n = 5

.0337 ± .9200i

.0804 ± 1.8451i

-1.5551 ± .1241i

-1.5652 ± .6569i

.0184 ± .0828i

.0194 ± .2531i

.0217 ± .4385i

.0260 ± .6530i

-1.5582 ± .3793i

-1.5780 ± .9781i

-1.6009 ± 1.3775i

-1.6447 ± 1.9204i

.0483 ± 1.2837i

.1725 ± 2.9036i

-1.7411 ± 2.7558i

-1.9974 ± 4.3007i

.6776 ± 5.8866i

-3.8575 ± 8.6059i

n = 6

.0151 ± .0684i

.0157 ± .2078i

.0170 ± .3553i

.0191 ± .5180i

.0226 ± .7056i

.0282 ± .9335i

-1.5454 ± .1025i

-1.5471 ± .3114i

-1.5509 ± .5326i

-1.5573 ± .7763i

.0378 ± 1.2285i

-1.5677 ± 1.0574i

.0558 ± 1.6430i

.2066 ± 3.5579i

.0948 ± 2.2989i

-1.6669 ± 2.4584i

-1.7843 ± 3.4350i

-1.5845 ± 1.3985i

-2.0951 ± 5.2580i

-4.3888 ± 10.4868i

.8331 ± 7.1412i

n = 7

.0129 ± .0582i

.0132 ± .1763i

.0140 ± .2993i

.0152 ± .4313i

.0171 ± .5772i

.0199 ± .7443i

-1.5386 ± .0873i

-1.5397 ± .2643i

-1.5419 ± .4487i

-1.5457 ± .6465i

.0243 ± .9432i

.0637 ± 1.9895i

.0425 ± 1.5196i

.0311 ± 1.1913i

.1096 ± 2.7436i

.2411 ± 4.2065i

-1.5513 ± .8653i

-1.5931 ± 1.7848i

-1.5727 ± 1.4134i

-1.5598 ± 1.1156i

-1.6273 ± 2.2757i

1.6906 ± 2.9769i

2.1961 ± 6.2031i

1.8282 ± 4.1012i

1.9236 ± 12.3664i

.9929 ± 8.3935i

<u>n = 8</u>

.0112 ± 0507i

.0114 ± .1532i

.0119 ± .2589i

.0127 ± .3703i

.0139 ± .4904i

.0155 ± .6232i

.0213 ± .9503i

.0342 ± 1.4362i

.0718 ± 2.3284i

.2757 ± 4.8518i

-1.5336 ± .0760i

-1.5358 ± .3882i

-1.5416 ± .7353i

-1.5536 ± 1.1602i

-1.5790 ± 1.7452i

-1.6422 ± 2.6968i

-1.8719 ± 4.7598i

-2.3000 ± 7.1402i

.0263 ± 1.1645i

.0474 ± 1.8007i

.1246 ± 3.1828i

1.1560 ± 9.6446i

-1.5343 ± .2297i

-1.5381 ± .5552i

-1.5466 ± .9343i

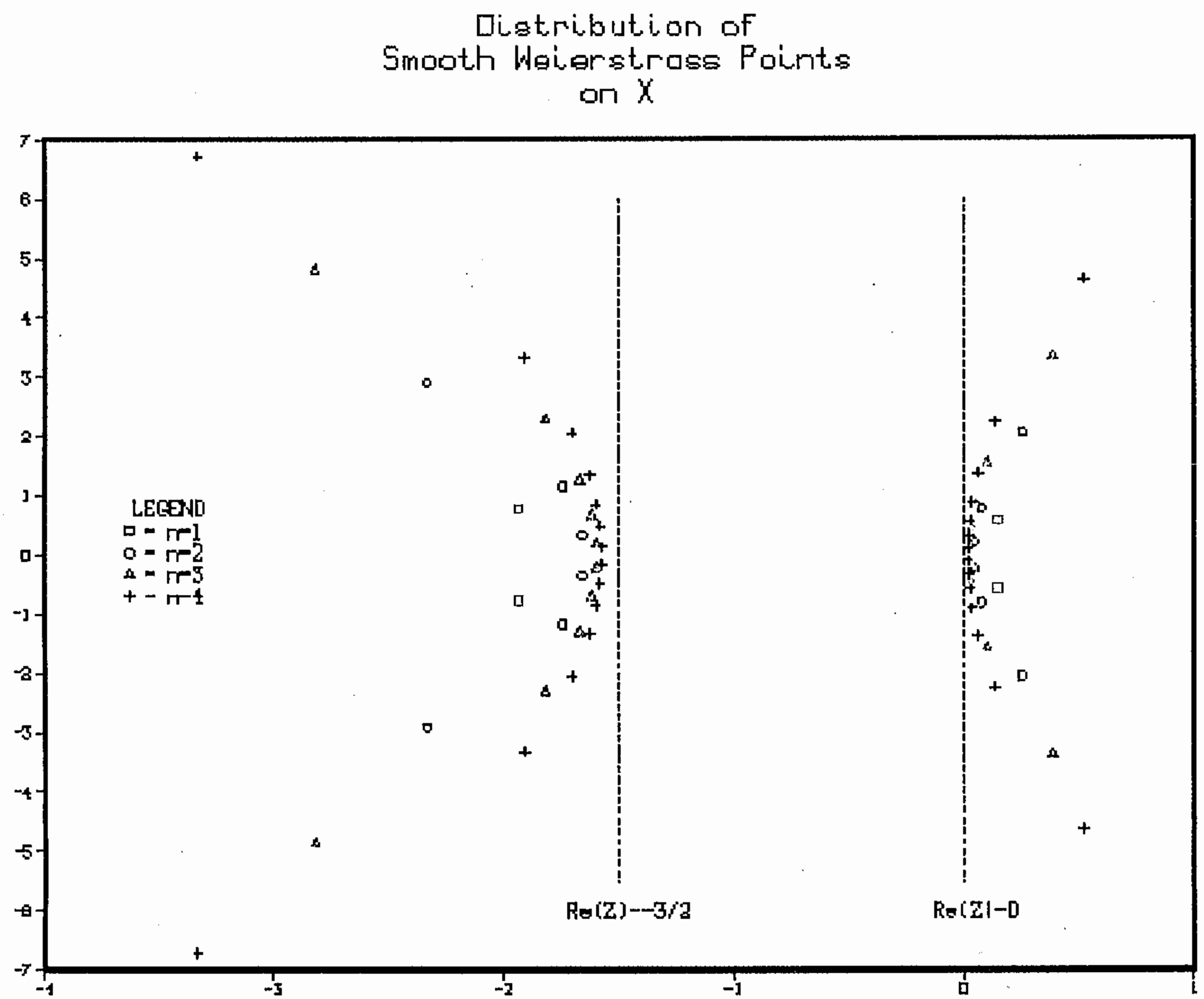
-1.5639 ± 1.4245i

-1.6026 ± 2.1518i

-1.7150 ± 3.4839i

-5.4607 ± 14.2452i

.0179 ± .7739i


Distribution of
Smooth Weierstrass Points
on X
LEGEND
□ = n=1
○ = n=2
△ = n=3
+ - n=4
Re(Z)=-3/2
Re(Z)=0
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-4
-3
-2
-1
0
1

Appendix B
Smooth Weierstrass Points for X at Infinity

n = 1

4711 ± .6960i

5289 ± 3.6766i

n = 2

6060 ± .9517i

6987 ± 9.1377i

8381 ± 1.1921i

1347 ± .3469i

5966 ± 1.0547i

1259 ± 3.0544i

n = 3

.2590 ± .6646i

.2917 ± 1.1657i

.0738 ± .2274i

.7190 ± 1.5970i

.6584 ± 1.0244i

.0190 ± 14.5347i

.3947 ± 5.3510i

.0227 ± .8491i

.8648 ± 2.8588i

.6970 ± .7226i

n = 4

.3550 ± .8084i

.1473 ± 1.1865i

.0500 ± .1689i

.6850 ± 7.5258i

.9588 ± 4.2934i

.3562 ± 19.8950i

.6764 ± 1.8106i

.6854 ± 1.0587i

.1484 ± .5006i

.7572 ± 2.7633i

.7016 ± 1.0904i

2.6352 ± 1.1121i

2.2020 ± .6799i

2.6415 ± .5193i

n = 5

.7019 ± 1.0786i

1.4959 ± 1.1723i

13.6994 ± 25.2420i

3.0611 ± 5.6528i

.0376 ± .1343i

.0986 ± .4001i

.2246 ± .6556i

.4230 ± .8888i

3.9820 ± 9.6589i

2.8038 ± 3.8096i

2.6984 ± 2.7079i

2.6463 ± 1.9417i

1.0646 ± 1.1900i

1.9414 ± .9697i

2.6181 ± 1.3504i

2.5995 ± .8604i

2.2927 ± .5604i

2.6110 ± .4053i

n = 6

.0300 ± .1115i

.0715 ± .3329i

.1565 ± .5489i

.2886 ± .7541i

.4725 ± .9396i

.7130 ± 1.0916i

16.0454 ± 30.5828i

4.2821 ± 11.7229i

3.1668 ± 6.9778i

2.8547 ± 4.8042i

1.0114 ± 1.1891i

2.7261 ± 3.5320i

1.3605 ± 1.2034i

2.0890 ± .8568i

1.7361 ± 1.1012i

2.6031 ± 1.5184i

2.5897 ± 1.0843i

2.6615 ± 2.6716i

2.5797 ± .7042i

2.5916 ± .3324i

2.3450 ± .4745i

n = 7

.0249 ± .0953i

.0550 ± .2848i

.1163 ± .4713i

.2106 ± .6516i

.3408 ± .8215i

.5100 ± .9745i

18.3931 ± 35.9197i

4.5838 ± 13.8749i

3.2744 ± 8.2842i

2.9076 ± 5.7714i

.7209 ± 1.1008i

1.5843 ± 1.1651i

1.2663 ± 1.2154i

.9743 ± 1.1871i

1.9035 ± 1.0168i

2.1850 ± .7611i

2.7562 ± 4.3187i

2.6093 ± 2.0954i

2.6362 ± 2.6461i

.6797 ± 3.3519i

.5919 ± 1.6427i

.5803 ± 1.2546i

.5670 ± .5970i

.5727 ± .9085i

.5780 ± .2817i

.3781 ± .4106i

<u>n = 8</u>

.0213 ± .0832i

.0441 ± .2489i

.0904 ± .4126i

.1612 ± .5725i

.2582 ± .7263i

.5392 ± .9997i

.7269 ± 1.1076i

.9471 ± 1.1848i

1.1973 ± 1.2194i

1.4706 ± 1.1979i

1.7531 ± 1.1064i

2.0229 ± .9347i

2.2505 ± .6816i

2.4005 ± .3614i

20.7417 ± 41.2545i

4.8865 ± 15.9709i

3.3830 ± 9.5793i

2.9617 ± 6.7223i

2.6995 ± 4.0037i

2.6490 ± 3.2254i

2.6177 ± 2.6272i

2.5970 ± 2.1443i

2.5829 ± 1.7388i

2.5731 ± 1.3867i

2.5662 ± 1.0723i

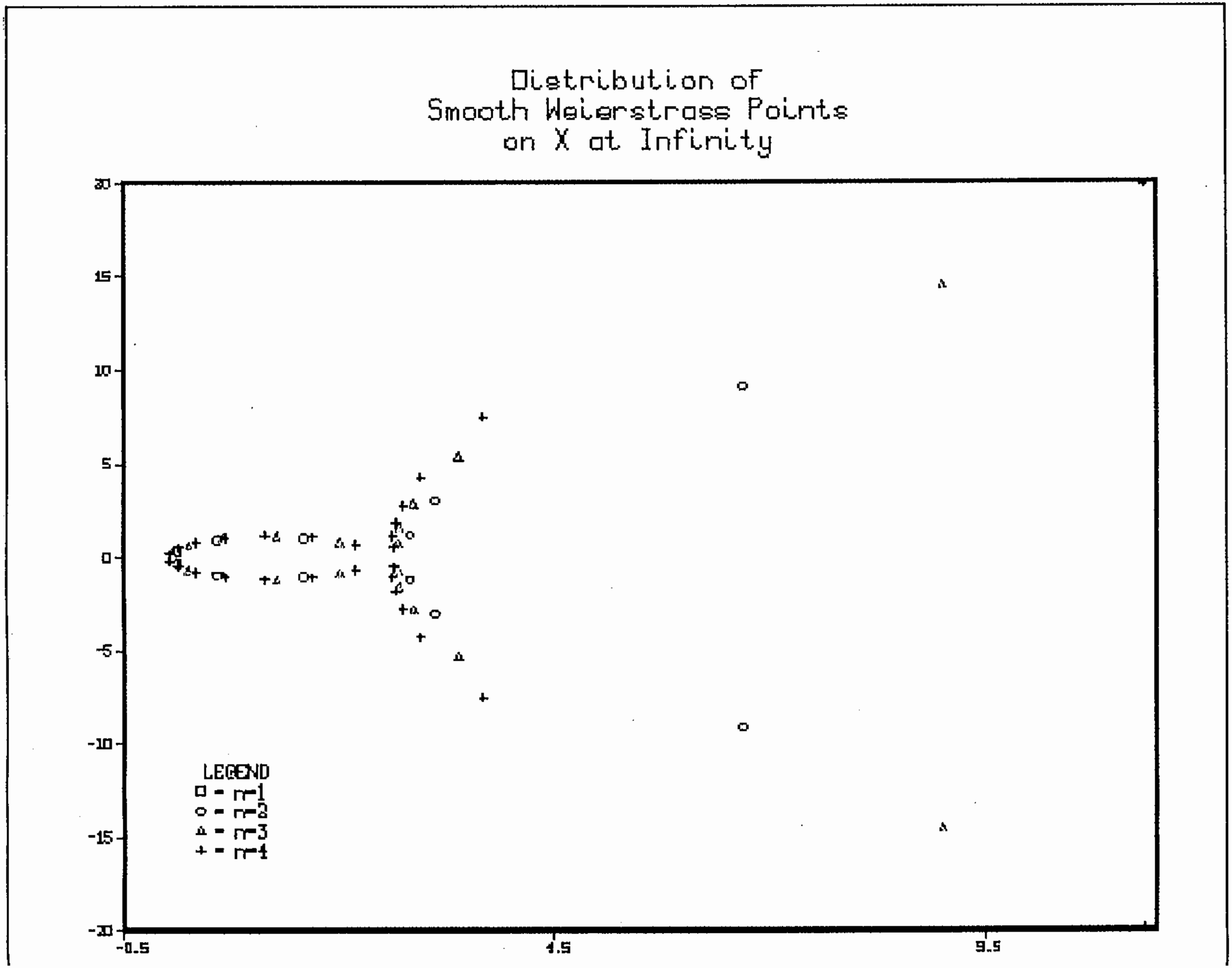
2.5612 ± .7831i

2.5679 ± .2444i

2.5581 ± .5187i

2.7876 ± 5.0835i

.3835 ± .8703i


Distribution of
Smooth Weierstrass Points
on X at Infinity
20
15
10
5
0
-5
-10
-15
-20
-0.5
4.5
9.5
LEGEND
□ = n=1
○ = n=2
△ = n=3
+ = n=4

```
LOATING,ENVIRONMENT('REALARITH.PEN')]
ram CTransform(Input,Output);
{This program performs the calculation of the linear fractional
 transformation which sends -3/2 to infinity, 0 to 0 and
 1 to 1.

 Written by:    Kathryn Furio
 Date:          March 28, 1988
 Revised:       April 7, 1988}



Complex = RECORD
   Re, Im:Double;
End;


Z, sl, s2, answer, numb: complex;
response:char;

LUDE 'mcpack.pas'

n  {program}

.Re:=0;
.Im:=1;
AT
iteln;
iteln('Enter the number to be transformed.');
iteln('Real part of Z: ');
adln(Z.Re);
iteln;
iteln('Imaginary part of Z:');
adln(Z.Im);
iteln;
:=CMult(Z,C(5));
:=CAdd(C(3),CMult(C(2),Z));
swer:=CMult(sl,CInverse(s2));
iteln('The answer is ', answer.Re:1:7, '+', answer.Im:1:7,'i');
iteln;
iteln('Would you like to try another value of Z?');
adln(response);
L response in ['N','n'];

  {program}
```

Appendix C

Smooth Weierstrass Points for Example 3

n = 1

.4387 + .2888i
.5000 - .4998i
1.2279 + .7112i
.4999 - .5002i

n = 2

.4595 + .3968i
1.3807 + 1.0855i
1.1067 + .5877i
.5000 - .4999i
.5021 - .5138i
.6968 - .7540i
1.9112 + 2.0044i
.4999 - .5001i
.4733 + .1159i
.8810 + .1233i
.4553 - .8654i

n = 3

.4724 + .4325i
.4811 + .2280i
.4463 + .7754i
.4836 + .0716i
1.4243 + 1.2230i
.9243 + .2662i
.5000 - .5001i
.4855 - .2258i
.6073 - 1.4835i
.5000 - .4999i
.4776 - .6926i
.7682 - .0658i
.4832 - .0705i
2.5111 + 3.0404i
1.2222 + .8548i
.4890 - .4812i
1.7531 + 1.7791i
1.0674 + .5556i
-.1121 - 1.4500i

n = 4

.4791 + .4498i
.4666 + .6871i
.4872 + .1624i
.4846 + .2885i
.4884 + .0525i
1.2836 + .9946i
1.4426 + 1.2903i
.9445 + .3299i
1.0494 + .5408i
.5000 - .4999i
.5000 - .5001i
.4881 - .0528i
.4850 - .6311i
.4865 - .1614i
.4879 - .2864i
.7084 - .1653i
.5422 - 1.0533i
.6204 - 2.0832i
1.6768 + 1.7047i
1.1587 + .7554i
.8347 + .1031i
-.5354 - 2.2282i
.1382 - 1.1041i
2.0923 + 2.3899i
3.0961 + 4.0342i

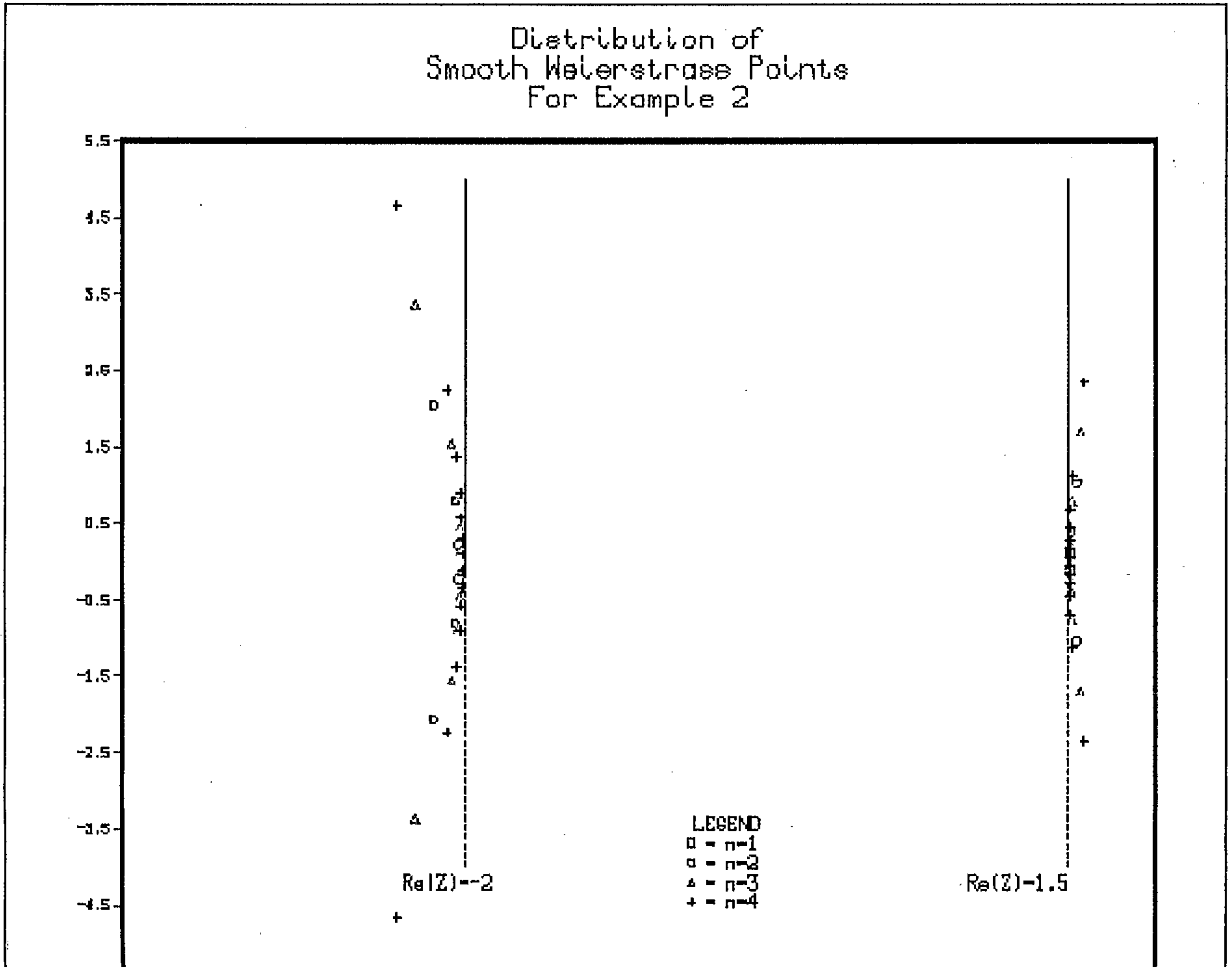
n = 5

.4903 + .1266i
.4891 + .2192i
.4870 + .3265i
1.4545 + 1.3308i
1.2160 + .8795i
1.3225 + 1.0832i
.4896 - .3246i
-.1237 - 1.6279i
1.9097 + 2.1425i
.4901 - .1268i
.4884 - .2186i
.5000 - .4999i
.4832 + .4600i
.4758 + .6418i
.4908 + .0414i
.9562 + .3661i
1.0389 + .5322i
.4909 - .0415i
.5000 - .5001i
.4888 - .5994i
.5233 - .8857i
.5360 - 1.3667i
1.6333 + 1.6583i
1.1238 + .7001i
.2395 - .9466i
2.4201 + 2.9739i
3.6760 + 5.0129i
.8711 + .1925i

n = 6

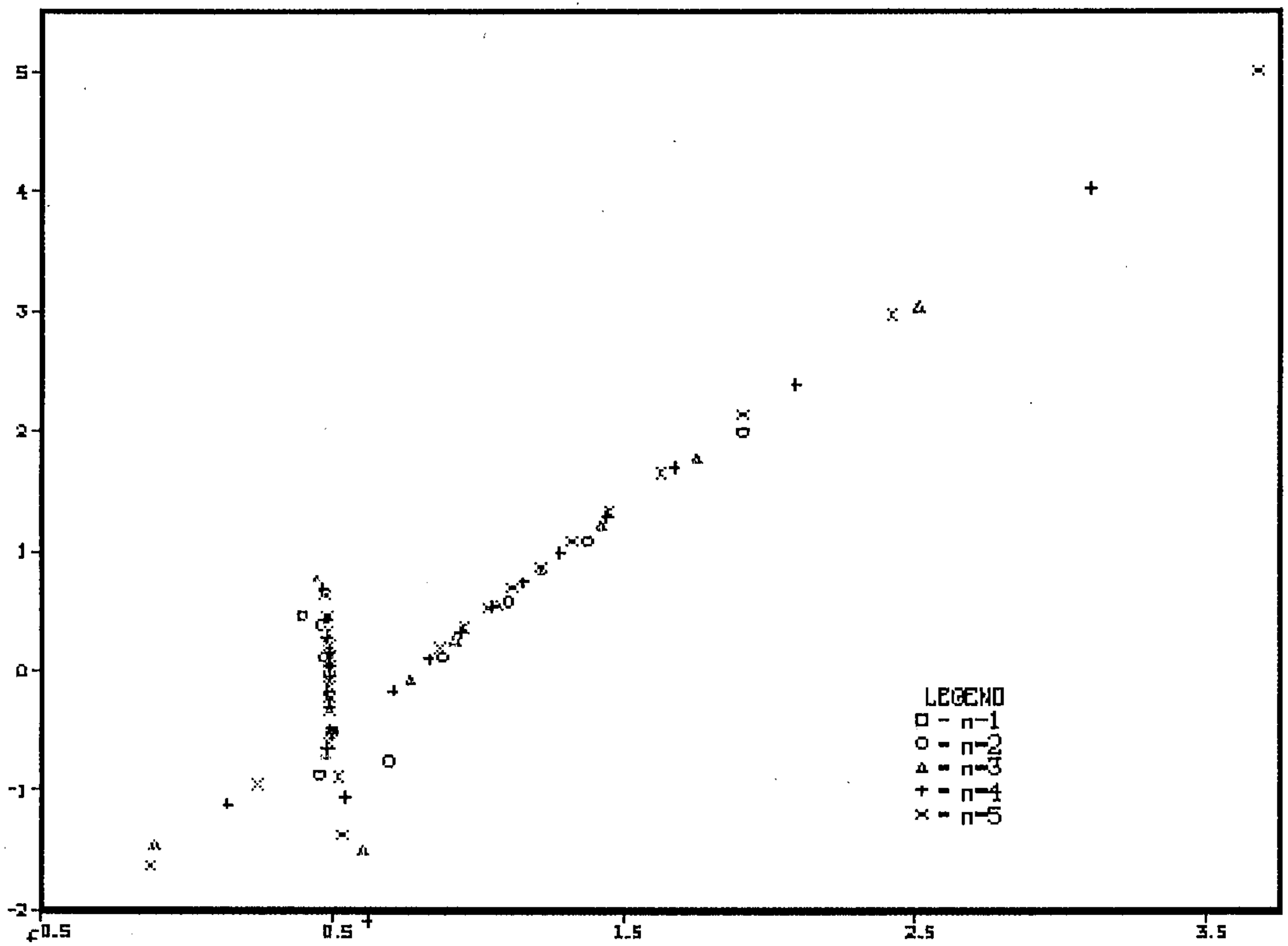
.4921 + .1039i
.4915 + .1777i

.4904 + .2590i	2.7425 + 3.5458i	1.2861 + 1.0308i
1.4623 + 1.3583i	1.3494 + 1.1449i	.4914 + .2886i
1.1752 + .8090i	.5443 - 1.6723i	1.3691 + 1.1905i
1.2562 + .9662i	.8203 + .0970i	.9102 + .2869i
.4897 - .2585i	-.3879 - 2.1402i	3.0621 + 4.1109i
.0419 - 1.3357i	.4887 + .3528i	.8486 + .1616i
1.8114 + 1.9988i	.8941 + .2484i	.4900 + .3721i
.4922 - .1039i	.7388 - .0720i	.4788 + .7598i
.4914 - .1779i	.4934 + .0881i	.4923 - .2159i
.5000 - .4999i	.4930 + .1497i	.1380 - 1.1632i
.4859 + .4668i	.4924 + .2156i	.4934 - .0881i
.4811 + .6142i	1.4678 + 1.3782i	.4931 - .1497i
.4924 + .0342i	1.1477 + .7610i	.5000 - .4999i
.9638 + .3896i	1.2135 + .8898i	.4936 - .0291i
1.0322 + .5266i	1.7527 + 1.9078i	.5000 - .5001i
.4924 - .0342i	.4879 + .4716i	.4925 - .5670i
.5000 - .5001i	.4845 + .5956i	.5113 - .7407i
.4910 - .5801i	.4936 + .0291i	.5135 - .9595i
.5154 - .7963i	.9692 + .4061i	.3314 - .7939i
.5195 - 1.0991i	1.0274 + .5227i	.5255 - 1.3127i
1.6079 + 1.6287i	1.5907 + 1.6087i	-.1532 - 1.7099i
1.1017 + .6646i	1.0863 + .6399i	.4908 - .2882i
.2955 - .8545i	1.9831 + 2.3251i	.5533 - 1.9787i
2.1337 + 2.5611i	2.3531 + 2.9694i	



Distribution of
Smooth Weierstrass Points
For Example 2
5.5
4.5
3.5
2.5
1.5
0.5
-0.5
-1.5
-2.5
-3.5
-4.5
Re(Z)=-2
LEGEND
□ = n=1
○ = n=2
△ = n=3
+ = n=4
Re(Z)=1.5

Distribution of
Smooth Weierstrass Points
For Example 3



Appendix D
Smooth Weierstrass Points for Example 2

n = 1

1.5226 ± .2879i

-2.0941 ± .5698i

n = 2

1.5129 ± .3985i

1.5421 ± 1.0364i

1.5083 ± .1141i

-2.0369 ± .2277i

-2.1835 ± 2.0524i

-2.0572 ± .7947i

n = 3

1.5061 ± .2283i

1.5089 ± .4332i

1.5052 ± .0719i

1.5640 ± 1.7005i

1.5173 ± .7777i

-2.0234 ± .1437i

-2.0402 ± .8653i

-2.0277 ± .4563i

-2.0786 ± 1.5517i

-2.2886 ± 3.3678i

1.5050 ± .2887i

1.5041 ± .1624i

1.5067 ± .4502i

1.5108 ± .6881i

1.5225 ± 1.1226i

1.5860 ± 2.3494i

1.5038 ± .0525i

-2.0172 ± .1051i

-2.0188 ± .3248i

-2.0307 ± .8998i

-2.0226 ± .5771i

-2.0492 ± 1.3749i

-2.1031 ± 2.2408i

-2.3972 ± 4.6544i

n = 5

1.5031 ± .1266i

1.5035 ± .2193i

1.5042 ± .3267i

1.5054 ± .4603i

1.5130 ± .9238i

1.5078 ± .6423i

1.5278 ± 1.4560i

1.6082 ± 2.9926i

1.5030 ± .0414i

-2.0135 ± .0828i

-2.0143 ± .2532i

-2.0160 ± .4385i

-2.0192 ± .6531i

-2.0248 ± .9202i

-2.0356 ± 1.2841i

-2.0594 ± 1.8462i

-2.5076 ± 5.9310i

-2.1284 ± 2.9071i

n = 6

1.5025 ± .1039i

1.5027 ± .1777i

1.5031 ± .2591i

1.5036 ± .3529i

1.5045 ± .4669i

1.5061 ± .6145i

1.5090 ± .8221i

1.5153 ± 1.1509i

1.5333 ± 1.7839i

1.5024 ± .0342i

1.6305 ± 3.6332i

-2.0112 ± .0684i

-2.0116 ± .2078i

-2.0166 ± .7057i

-2.0125 ± .3553i

-2.0141 ± .5181i

-2.0208 ± .9337i

-2.0279 ± 1.2288i

-2.0412 ± 1.6436i

-2.0701 ± 2.300i

-2.6191 ± 7.2031i

n = 7

1.5021 ± .0882i

1.5025 ± .2157i

1.5028 ± .2887i

1.5032 ± .3722i

1.5039 ± .4717i

1.5050 ± .5959i

1.5068 ± .7602i

1.5103 ± .9955i

1.5177 ± 1.3735i

1.5021 ± .0291i

1.6530 ± 4.2722i

-2.0095 ± .0582i

-2.0098 ± .1763i

-2.0103 ± .2994i

-2.0112 ± .4313i

-2.0126 ± .5773i

.0147 ± .7444i

.0179 ± .9433i

.0229 ± 1.1915i

.0313 ± 1.5200i

.0470 ± 1.9903i

.0810 ± 2.7455i

.7315 ± 8.4726i

<u>n = 8</u>

.5020 ± .1852i

.5025 ± .3116i

1.5029 ± .3870i

1.5034 ± .4753i

1.5042 ± .5824i

1.5055 ± .7183i

1.5076 ± .9008i

1.5116 ± 1.1650i

1.5201 ± 1.5934i

1.6756 ± 4.9105i

1.5018 ± .0766i

-2.0088 ± .2589i

-2.0094 ± .3703i

-2.0102 ± .4905i

-2.0114 ± .6232i

-2.0132 ± .7740i

-2.0157 ± .9504i

-2.0194 ± 1.1647i

-2.0252 ± 1.4364i

-2.0350 ± 1.8011i

-2.0530 ± 2.3293i

```
his is a package of definitions and basic
outines for complex arithmetic for use in New
atrixcal and other programs.  The type Complex
      is defined as:

ype
Complex = Record
 Re,Im: Real;
End;

n the main program.}

unction C(X:DOUBLE): Complex;

Converts X to the Complex number X+0i}

Var
 Temp: Complex;

Begin
Temp.Re:=X;
Temp.Im:=0.0;
C:=Temp;
End;  {C}

unction CAdd(Z1,Z2: Complex): Complex;

Adds the complex numbers Z1 and Z2}

Var
 Temp: Complex;

Begin
Temp.Re:=Z1.Re + Z2.Re;
Temp.Im:=Z1.Im + Z2.Im;
CAdd:=Temp;
End;  {CAdd}

unction CSub(Z1,Z2: Complex): Complex;

Subtracts the complex number Z2 from Z1}

Var
 Temp: Complex;

Begin
Temp.Re:=Z1.Re - Z2.Re;
Temp.Im:=Z1.Im - Z2.Im;
CSub:=Temp;
End;  {CSub}

unction CMult(Z1,Z2: Complex): Complex;

Multiplies Z1 and Z2}

Var
 Temp: Complex;

Begin
Temp.Re:=Z1.Re * Z2.Re - Z1.Im * Z2.Im;
Temp.Im:=Z1.Re * Z2.Im + Z1.Im * Z2.Re;
CMult:=Temp;
End;  {CMult}

unction CConj(Z: Complex): Complex;
```

```
;omputes the complex conjugate of Z}

'ar
 Temp: Complex;

}egin
'emp.Re:=Z.Re;
'emp.Im:=-Z.Im;
:Conj:=Temp;
:nd;  {CConj}

inction CAbs(Z: Complex): DOUBLE;

:omputes the absolute value of Z}

iegin
:Abs:=Sqrt(Sqr(Z.Re) + Sqr(Z.Im));
:nd;  {CAbs}

.nction CInverse(Z: Complex): Complex;

 nverts Z, assuming Z <> 0}

iegin
:Inverse:=CMult(CConj(Z),C(1.0D0/Sqr(CAbs(Z))));
:nd;  {CInverse}

              Function CPower(Z:Complex; p:integer):Complex;

              {Computes Z to the pth power}

                     Var r, theta:DOUBLE;
                         temp:Complex;
                         ratio:DOUBLE;

                     Begin
                     ratio:=Z.Im/Z.Re;
                     r:=Sqrt(Sqr(Z.Re) + Sqr(Z.Im));
                     if Z.Re>0 then theta:= arctan(ratio)
                     else theta:= arctan(ratio) + 3.14159265;
                     temp.Re:= (r**p)*(cos(p*theta));
                     temp.Im:= (r**p)*(sin(p*theta));
                     CPower:=temp;
                     End;  {CPower}
```

```
.......,.........( .........  )]
GRAM ComplexNewton(Input,Output);

is program applies the usual (one variable) Newton's method for finding
ots of equations  f(x) = 0  to solving polynomial equations
th complex coefficients.  The routines for complex arithmetic
e contained in the %INCLUDE file 'mcpack.pas'.

itten by: John Little
te:         Feb 2, 1988
vised by: Kathryn Furio
te:         March 20, 1988}

CONST
   Tolerance = 1.0D-8;

TYPE
   Complex = RECORD
      Re,Im: Double;
   END;

VAR
   Starting, Next, A, B: Complex;
   Difference: Double;
   n, p, Iterations: Integer;
   Response, Response2: Char;
   b1, c1, b2, c2, s1, s2, s3, s4:complex;

   %INCLUDE 'mcpack.pas'

   FUNCTION EvalF(p: Integer; b1,b2,c1,c2,s1,s2,s3,s4,Z: Complex): Complex;


      VAR
        P1, P2, P3, P4, sum1, sum2:complex;

      BEGIN  {EvalF}
      P1:=CMult(s1,CMult(CPower(CSub(b2,Z),p),CPower(CSub(b1,Z),p)));
      P2:=CMult(s2,CMult(CPower(CSub(c2,Z),p),CPower(CSub(b1,Z),p)));
      P3:=CMult(s3,CMult(CPower(CSub(c1,Z),p),CPower(CSub(b2,Z),p)));
      P4:=CMult(s4,CMult(CPower(CSub(c1,Z),p),CPower(CSub(c2,Z),p)));
      sum1:=CAdd(P1,P2);
      sum2:=CAdd(P3,P4);
      EvalF:=CAdd(sum1, sum2);
      END; {EvalF}

   FUNCTION EvalDer(p:integer;b1,b2,c1,c2,s1,s2,s3,s4,Z:Complex):Complex;

      VAR
         P1, P2, P3, P4, P5, P6, P7, P8,sum1, sum2:complex;

      BEGIN  {EvalDer}
      P1:=CMult(C(p),CMult(CPower(CSub(b2,Z),p-1),CPower(CSub(b1,Z),p)));
      P2:=CMult(C(p),CMult(CPower(CSub(b2,Z),p),CPower(CSub(b1,Z),p-1)));
      P3:=CMult(C(p),CMult(CPower(CSub(c2,Z),p-1),CPower(CSub(b1,Z),p)));
      P4:=CMult(C(p),CMult(CPower(CSub(c2,Z),p),CPower(CSub(b1,Z),p-1)));
      P5:=CMult(C(p),CMult(CPower(CSub(c1,Z),p-1),CPower(CSub(b2,Z),p)));
      P6:=CMult(C(p),CMult(CPower(CSub(c1,Z),p),CPower(CSub(b2,Z),p-1)));
      P7:=CMult(C(p),CMult(CPower(CSub(c1,Z),p-1),CPower(CSub(c2,Z),p)));
      P8:=CMult(C(p),CMult(CPower(CSub(c1,Z),p),CPower(CSub(c2,Z),p-1)));
      sum1:=CAdd(CAdd(CMult(s1,P1),CMult(s1,P2)),CAdd(CMult(s2,P3),
            CMult(s2,P4)));
      sum2:=CAdd(CAdd(CMult(s3,P5),CMult(s3,P6)),CAdd(CMult(s4,P7),
            CMult(s4,P8)));
      EvalDer:=CMult(C(-1),CAdd(sum1,sum2));
      END;  {EvalDer}
```

```
BEGIN {Main}
writeln;
writeln('Enter the values of the points being identified in C.');
writeln('Enter these in the order bl, cl, b2, c2, where bl and');
writeln('cl are identified, b2 and c2 are identified.');
writeln;
writeln('Real Part of bl:');
readln(bl.Re);
writeln;
writeln('Imaginary Part of bl:');
readln(bl.Im);
writeln;
writeln('Real Part of cl:');
readln(cl.Re);
writeln;
writeln('Imaginary Part of cl:');
readln(cl.Im);
writeln;
writeln('Real Part of b2:');
readln(b2.Re);
writeln;
writeln('Imaginary Part of b2:');
readln(b2.Im);
writeln;
writeln('Real Part of c2:');
readln(c2.Re);
writeln;
writeln('Imaginary Part of c2:');
readln(c2.Im);
writeln;
sl:=CSub(b2,bl);
s2:=CSub(bl,c2);
s3:=CSub(cl,b2);
s4:=CSub(c2,cl);
REPEAT
writeln('Enter the value of n.');
writeln;
readln(n);
p:=4*n-1;
REPEAT
   Iterations:=0;
   Writeln('Enter first approximation to root');
   Writeln('Real part:');
   Readln(Starting.Re);
   Writeln('Imaginary Part:');
   Readln(Starting.Im);
   Writeln;
   REPEAT
      Iterations:=Iterations+1;
      A:=EvalF(p,bl,b2,cl,c2,sl,s2,s3,s4,Starting);
      B:=EvalDer(p,bl,b2,cl,c2,sl,s2,s3,s4,Starting);
      Next:=CSub(Starting,CMult(A,CInverse(B)));
      Writeln(Iterations:1,'th approximation to root: ');
      Writeln(Next.Re:1:7,'+',Next.Im:1:7,'i');
      Difference:=CAbs(CSub(Starting,Next));
      Starting:=Next;
   UNTIL (Iterations = 20) OR (Difference < Tolerance);
   Writeln('Do you want to try again with another starting value (Y/N)?');
   Readln(Response);
UNTIL Response IN ['N','n'];
writeln('Would you like to try another value of n?');
readln(Response2);
writeln;
UNTIL Response2 in ['N', 'n'];
END. {Main}
```