A Characterization of Metric Spheres in Hyperbolic Space by Morse Theory

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A CHARACTERIZATION OF METRIC SPHERES IN HYPERBOLIC SPACE BY MORSE THEORY

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0. Introduction. Let M^n be a differentiable manifold of class C^∞. By a Morse function f on M^n, we mean a differentiable function f on M^n having only non-degenerate critical points. A well-known topological result of Reeb states that if M^n is compact and there is a Morse function f on M^n having exactly 2 critical points, then M^n is homeomorphic to an n-sphere, S^n (see, for example, [3], p. 25).

In a recent paper, [4], Nomizu and Rodriguez found a geometric characterization of a Euclidean n-sphere $S^n \subset R^{n+p}$ in terms of the critical point behavior of a certain class of functions L_p, $p \in R^{n+p}$, on M^n. In that case, if $p \in R^{n+p}$, $x \in M^n$, then $L_p(x) = (d(x, p))^2$, where d is the Euclidean distance function.

Nomizu and Rodriguez proved that if M^n $(n \geqq 2)$ is a connected, complete Riemannian manifold isometrically immersed in R^{n+p} such that every Morse function of the form L_p, $p \in R^{n+p}$, has index 0 or n at any of its critical points, then M^n is embedded as a Euclidean subspace, R^n, or a Euclidean n-sphere, S^n. This result includes the following: if M^n is compact such that every Morse function of the form L_p has exactly 2 critical points, then $M^n = S^n$.

In this paper, we prove results analogous to those of Nomizu and Rodriguez for a submanifold M^n of hyperbolic space, H^{n+p}, the space-form of constant sectional curvature -1.

For $p \in H^{n+p}$, $x \in M^n$, we define the function $L_p(x)$ to be the distance in H^{n+p} from p to x. We then define the concept of a focal point of (M^n, x) and prove an Index Theorem for L_p which states that the index of L_p at a non-degenerate critical point x is equal to the number of focal points of (M^n, x) on the geodesic in H^{n+p} from x to p.

In section 2, we prove that a metric sphere $S^n \subset H^{n+p}$ can be characterized by the condition that every Morse function of the form L_p, $p \in H^{n+p}$, has exactly 2 critical points.

In section 3, we give an example which shows that a result analo-

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gous to that of Nomizu and Rodriguez for the non-compact case cannot be proven. More explicitly, we exhibit a complete surface $M^2 \subset H^3$ which is not umbilic on which every Morse function of the type L_p has index 0 at any of its critical points.

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1. The functions L_p and the index theorem. We will use the following representation of hyperbolic space H^m (for more detail, see [2], vol. II, p. 268). Consider R^{m+1} with a natural basis $e_0, e_1, \cdots, e_m$ and a non-degenerate quadratic form H defined by

$$H(x, y) = -x^0y^0 + \sum_{k=1}^{m} x^k y^k \quad \text{for} \quad x = \sum_{k=0}^{m} x^k e_k \quad \text{and} \quad y = \sum_{k=0}^{m} y^k e_k .$$

Then H^m is the hypersurface

$$\{x \in R^{m+1} \mid H(x, x) = -1, x^0 \geqq 1\} ,$$

on which g, the restriction of H, is a positive definite metric of constant sectional curvature -1.

Let M^n be a connected, Riemannian manifold, and let f be an isometric immersion of M^n into H^{n+p}. We first define the following class of functions on H^{n+p}; for p, q in H^{n+p}

$$L_p(q) \equiv d(p, q) ,$$

the distance in H^{n+p} from p to q. If we use the above representation of H^{n+p}, then we have

$$L_p(q) = \cosh^{-1}(-H(p, q)) .$$

For $p \in H^{n+p}$, $x \in M^n$, we define $L_p(x) = L_p(f(x))$. If $p \notin f(M^n)$, then the restriction of L_p to M^n is a differentiable function on M^n. From this point on, we will only consider L_p such that $p \notin f(M^n)$.

We now proceed to develop the concept of focal point and prove an Index Theorem for L_p. Let $N(M^n)$ denote the normal bundle of M^n. Any point of $N(M^n)$ can be represented as $(u, r\xi)$ where $u \in M^n$, $r \in R$, and ξ is a unit length vector in $T_u{}^{\perp}(M^n)$, the normal space to M^n at u.

We define $\gamma(u, \xi, r)$, $-\infty < r < \infty$, to be the geodesic in H^{n+p} parametrized by arc-length parameter r such that

$$\gamma(u, \xi, 0) = u \quad \text{and} \quad \vec{\gamma}(u, \xi, 0) = \xi .$$

Let U be a local co-ordinate neighborhood of M^n with co-ordinates $u^1, \cdots, u^n$. Then, in terms of the co-ordinates $x^0, \cdots, x^{n+p}$ in R^{n+p+1}, the immersion $f(U)$ can be represented by the vector-valued function

$$x(u^1, \cdots, u^n) = (x^0(u^1, \cdots, u^n), \cdots, x^{n+p}(u^1, \cdots, u^n)) .$$

In terms of this representation, the geodesic $\gamma(u, \xi, r)$ is given by

$$\gamma(u, \xi, r) = (\cosh r)x(u) + (\sinh r)\xi .$$

We define a map F from $N(M^n)$ to H^{n+p} by

$$F(u, r\xi) = \gamma(u, \xi, r) .$$

As in the Euclidean case, the concept of focal point is defined in terms of the degeneracy of F_*, the Jacobian of F.

DEFINITION. A point $p \in H^{n+p}$ is called a *focal point of* (M^n, u) *of multiplicity* ν if $p = F(u, r\xi)$ and F_* has nullity $\nu > 0$ at $(u, r\xi) \in N(M^n)$. (We say p is a focal point of M^n if p is a focal point of (M^n, u) for some $u \in M^n$.)

For $\xi \in T_u^{\perp}(M^n)$, A_ξ denotes the symmetric endomorphism of $T_u(M^n)$ corresponding to the second fundamental form of M^n at u in the direction of ξ. The following proposition identifies the focal points of M^n.

PROPOSITION 1. *A point* $p \in H^{n+p}$ *is a focal point of* (M^n, y) *of multiplicity* $\nu > 0$ *if and only if*

$$p = F(y, r\xi) \quad \text{and} \quad \coth r = k$$

where k *is an eigenvalue of* A_ξ *of multiplicity* ν.

PROOF. Fix $(y, r\xi) \in N(M^n)$, and let U be a co-ordinate chart of M^n with co-ordinates $u^1, \cdots, u^n$ such that $y \in U$. Then $N(U)$ can be considered as $U \times R^p$. We now examine the nullity of F_* at $(y, r\xi)$.

We first assume $r \neq 0$. Choose $\xi_1, \cdots, \xi_p$ orthonormal normal vector fields on U such that $\xi_1(y) = \xi$. Let $\beta \in T_u^{\perp}(U)$ for some $u \in U$. Then we can write

$$\beta = \mu\Big(\sqrt{1 - \sum_{j=2}^{p} (t^j)^2}\,\xi_1 + t^2\xi_2 + \cdots + t^p\xi_p\Big) \quad \text{where}$$

$$0 \leqq \mu < \infty \quad \text{and} \quad \sum_{j=2}^{p} (t^j)^2 \leqq 1 .$$

The t^j are the direction cosines of β, and $\mu = \|\beta\|$. The coordinates $(u^1, \cdots, u^n, \mu, t^2, \cdots, t^p)$ are local co-ordinates on $N(U)$. For any j, we compute from the definition of F that,

$$F_*\Big(\frac{\partial}{\partial t^j}\Big)\Big|_{(y, r\xi)} = \vec{\eta}(t^j)\Big|_{t^j=0}$$

where the curve $\eta(t^j)$ is defined by

$$\eta(t^j) = (\cosh r)x(y) + (\sinh r)(\sqrt{1-(t^j)^2}\xi_1(y) + t^j\xi_j(y)) .$$

Then,

$$\vec{\eta}(t^j)\Big|_{t^j=0} = (\sinh r)\xi_j(y) \neq 0 \text{ and thus, } F_*\Big(\frac{\partial}{\partial t^j}\Big)\Big|_{(y,r\xi)} \neq 0 .$$

Similarly,

$$F_*\Big(\frac{\partial}{\partial \mu}\Big)\Big|_{(y,r\xi)} = \vec{\eta}(\mu)\Big|_{\mu=r} \quad \text{where } \eta(\mu) = (\cosh \mu)x(y) + (\sinh \mu)\xi_1(y) .$$

Then

$$\vec{\eta}(\mu) = (\sinh \mu)x(y) + (\cosh \mu)\xi_1(y) \quad \text{and} \quad \|\vec{\eta}(\mu)\| = 1 \quad \text{for all} \quad \mu .$$

In particular,

$$\vec{\eta}(\mu)\Big|_{\mu=r} = F_*\Big(\frac{\partial}{\partial \mu}\Big)\Big|_{(y,r\xi)} \neq 0 .$$

In fact, the above calculations show that if

$$V = a_1\Big(\frac{\partial}{\partial \mu}\Big) + \sum_{j=2}^{p} a_j\Big(\frac{\partial}{\partial t^j}\Big) \in T_{(y,r\xi)}(N(U)) ,$$

then $F_*(V) = 0$ only if $V = 0$. If we let

$$X = \sum_{j=1}^{n} b_j\Big(\frac{\partial}{\partial u^j}\Big) \in T_{(y,r\xi)}(N(U)) ,$$

we shall soon compute $F_*(X)$. That computation and the above will show that

$$F_*(X + V) = 0 \quad \text{only if} \quad V = 0 .$$

(We remark that if $r = 0$, we must choose a slightly different co-ordinate system to obtain the same result.)

Thus to find a vector $X \in T_{(y,r\xi)}(N(U))$ such that $F_*(X)$ vanishes, we must concern ourselves with vectors of the form

$$X = \sum_{j=1}^{n} b_j\Big(\frac{\partial}{\partial u^j}\Big) .$$

It is convenient to let $Y \in T_y(U)$ such that

$$X = (Y, 0)$$

when we consider $T_{(y,r\xi)}(N(U))$ as $T_y(M^n) \oplus R^p$. To facilitate the calculation of $F_*(X)$, we assume that the vector field ξ_1 defined above has been chosen so that $\nabla_Y{}^\perp \xi_1 = 0$, where $\nabla^\perp$ is the connection in the normal bundle induced by $\tilde{\nabla}$, the covariant derivative in H^{n+p}. From the definition of F we compute using the vector representation,

$$(1)\quad \begin{aligned} F_*(X) &= F_*(Y, 0) = \tilde{\nabla}_Y((\cosh r)x + (\sinh r)\xi_1) \\ &= (\cosh r)\tilde{\nabla}_Y x + (\sinh r)\tilde{\nabla}_Y \xi_1 = (\cosh r)Y + (\sinh r)\tilde{\nabla}_Y \xi_1 \,. \end{aligned}$$

However,

$$\tilde{\nabla}_Y \xi_1 = -A_{\xi_1} Y + \nabla_Y{}^{\perp}\xi_1 \,.$$

Since we have chosen ξ_1 so that

$$\nabla_Y{}^{\perp}\xi_1 = 0 \quad \text{and} \quad \xi_1(y) = \xi \quad \text{we have} \quad \tilde{\nabla}_Y \xi_1 = -A_\xi Y \,.$$

Thus (1) becomes

$$F_*(X) = F_*(Y, 0) = (\cosh r)Y - (\sinh r)A_\xi Y \,,$$

and we see that $F_*(Y, 0)$ vanishes if and only if

$$\coth r = k \,,$$

where k is an eigenvalue of A_ξ and Y is an eigenvector of k. This shows that if $\coth r$ has multiplicity $\nu > 0$ as an eigenvalue of A_ξ, then there is a ν-dimensional subspace of $T_{(y,r\xi)}(N(U))$ on which F_* vanishes. Thus in that case, $p = F(y, r\xi)$ is a focal point of multiplicity ν. q.e.d.

Next for $p \in H^{n+p}$, we want to examine the critical points on M^n of the function L_p. We will find an expression for the index of L_p at a non-degenerate critical point y of L_p. This and Proposition 1 yield an Index Theorem for L_p which states that the index of L_p at y equals the number of focal points on the geodesic in H^{n+p} from $f(y)$ to p. The following proposition characterizes the critical points of L_p on M^n.

PROPOSITION 2. *Let $p \in H^{n+p}$ and $x_0 \in M^n$ such that $f(x_0) \neq p$.*

(i) *x_0 is a critical point of L_p if and only if $p = F(x_0, r\xi)$ for ξ a unit vector in $T_{x_0}{}^{\perp}(M^n)$.*

(ii) *x_0 is a degenerate critical point of L_p if and only if* $\coth r = k$ *for k an eigenvalue of A_ξ.*

(iii) *If x_0 is a non-degenerate critical point of L_p, then the index of L_p at x_0 is equal to the number of eigenvalues k_i of A_ξ such that*

$$k_i > \coth r \,.$$

Here each k_i is counted with its multiplicity.

PROOF. For $x \in M^n$ and U a sufficiently small neighborhood of x, we may identify U with its image $f(U) \subset H^{n+p}$. Then using the vector representation of L_p, we compute the derivative of L_p. Fix $x_0 \in M^n$, and let X be a differentiable vector field on U. Then

$$XL_p(x) = X\cosh^{-1}(-H(x, p)) \tag{2}$$
$$= \frac{-1}{(H(x, p)^2 - 1)^{1/2}} H(D_X x, p) = \frac{-1}{(H(x, p)^2 - 1)^{1/2}} H(X, p) ,$$

where D is the Euclidean covariant derivative in R^{n+p+1}.

For the fixed point $x_0 \in U$, there is a unique unit-length vector $\beta \in T_{x_0}(H^{n+p})$ such that

$$p = (\cosh r)x_0 + (\sinh r)\beta \quad \text{where} \quad r = L_p(x_0) . \tag{3}$$

From (2) and (3) we have

$$XL_p(x_0) = \frac{-1}{(H(x_0, p)^2 - 1)^{1/2}} (\sinh r)H(X, \beta) , \tag{4}$$

since $H(X, x_0) = 0$ because $X \in T_{x_0}(H^{n+p})$.

From (4) we see that x_0 is a critical point of L_p if and only if $H(X, \beta) = 0$ for all $X \in T_{x_0}(M^n)$; that is, if and only if $\beta \in T^{\perp}_{x_0}(M^n)$, and thus $p = F(x_0, r\beta)$. This proves (i).

Now let $p = F(x_0, r\xi)$; we calculate the Hessian of L_p at x_0. Let X, Y be differentiable vector fields on U. Then for $x \in U$, we have

$$XL_p(x) = \frac{-1}{(H(x, p)^2 - 1)^{1/2}} H(X, p) . \tag{2}$$

Then since $H(X_{x_0}, p) = 0$, we have

$$YXL_p(x_0) = \frac{-1}{(H(x, p)^2 - 1)^{1/2}} Y(H(X, p))\Big|_{x_0} \tag{5}$$
$$= \frac{-1}{(H(x, p)^2 - 1)^{1/2}} H(D_Y X, p)\Big|_{x_0} .$$

From knowledge of the embedding of H^{n+p} in R^{n+p+1}, we know that for $x \in U$,

$$D_Y X|_x = \tilde{\nabla}_Y X|_x + H(X, Y)x \tag{6}$$

and

$$\tilde{\nabla}_Y X = \nabla_Y X + \alpha(X, Y) \tag{7}$$

for $\alpha(X, Y)$ the second fundamental form of M^n in H^{n+p}, and for ∇ the covariant derivative in M^n. Now (3), (6), (7) yield

$$H(D_Y X, p)|_{x_0} = \sinh rH(\alpha(X, Y), \xi) - \cosh rH(X, Y) \tag{8}$$
$$= \sinh rH(A_\xi X, Y) - \cosh rH(X, Y)$$
$$= H((\sinh rA_\xi - \cosh rI)X, Y)$$

where I is the identity endomorphism on $T_{x_0}(M^n)$.

We note that

$$H(x_0, p)^2 = \cosh^2 r \quad \text{and thus} \quad (H(x_0, p)^2 - 1)^{1/2} = \sinh r\ .$$

The above equation and (8) imply that we can re-write (5) as

$$YXL_p(x_0) = H((-A_\xi + \coth rI)X,\ Y)\,|_{x_0}\ . \tag{9}$$

From this expression for the terms of the Hessian of L_p at x_0, we conclude that x_0 is a degenerate critical point of L_p if and only if

$$\coth r = k$$

for k an eigenvalue of A_ξ, proving (ii).

The index of L_p at x_0 is defined as the number of negative eigenvalues of the Hessian of L_p at x_0. We see from (9) that if $\coth r$ is not an eigenvalue of A_ξ, then the index of L_p at x_0 equals the number of eigenvalues k_i of A_ξ, counted with their multiplicities, such that

$$k_i > \coth r\ .$$

This proves (iii) and completes the proof of Proposition 2. q.e.d.

Propositions 1 and 2 yield immediately the Index Theorem for L_p.

THEOREM 1. (Index Theorem for L_p) *For $p \in H^{n+p}$, the index of L_p at a non-degenerate critical point $x \in M^n$ is equal to the number of focal points of (M^n, x) which lie on the geodesic in H^{n+p} from $f(x)$ to p. Each focal point is counted with its multiplicity.*

2. A characterization of metric spheres in terms of the functions L_p.

We now proceed to prove the main result of this paper which we state as follows.

THEOREM 2. *Let M^n be a connected, compact, differentiable manifold immersed in H^{n+p}. If every Morse function of the form L_p, $p \in H^{n+p}$, has exactly 2 critical points, then M^n is embedded as a metric sphere, S^n.*

In the above statement, the notation "metric sphere" means the following. There exists a totally geodesic $(n+1)$-dimensional submanifold $H^{n+1} \subset H^{n+p}$, a point $q \in H^{n+1}$, and $c \in R$, such that

$$S^n = \{y \in H^{n+1} \mid d(q, y) = c\}\ .$$

In the remainder of this section, we assume M^n satisfies the hypotheses of Theorem 2. We first consider the set T,

$$T = \{p \in H^{n+p} \mid p \text{ is not a focal point of } M^n\}\ .$$

By Sard's Theorem, T is dense in H^{n+p} (see [3], p. 36). Propositions 1 and 2 show that L_p is a Morse function if and only if $p \in T$. Using these facts, we can prove the following proposition. With minor changes, the proof is identical to the proof of the corresponding proposition for submanifolds of R^m proven by Nomizu and Rodriguez ([4], p. 199). Hence, we omit the proof here.

PROPOSITION 3. *Let $p \in H^{n+p}$, and assume that L_p has a non-degenerate critical point at $x \in M^n$ of index j. Then, there is a point $q \in H^{n+p}$ such that L_q is a Morse function which has a critical point $z \in M^n$ of index j (q and z may be chosen as close to p and x, respectively, as desired).*

To prove Theorem 2 we will proceed in the following way. Let f be the immersion of M^n into H^{n+p}. We will show that f is umbilic. Then it is known that a compact umbilical submanifold of H^{n+p} must be a metric sphere S^n. The proof of this fact is very similar to Cartan's argument for submanifolds of R^m (see [1], p. 231).

We first prove the following result.

PROPOSITION 4. *Let $x \in M^n$ and suppose there is a unit length vector $\xi \in T_x^\perp(M^n)$ such that A_ξ has an eigenvalue whose absolute value is greater than 1. Then, $A_\xi = \lambda I$ for $\lambda \in R$.*

PROOF. Let λ be the eigenvalue of A_ξ with largest absolute value. We know from the hypothesis that

$$|\lambda| > 1 .$$

We may assume $\lambda > 1$; for if $\lambda < -1$, then we simply prove the proposition is true for $A_{-\xi}$ which has an eigenvalue $-\lambda > 1$. This will, of course, also prove the result for A_ξ.

Take $r > 0$ such that

$$\mu < \coth r < \lambda$$

where μ is the second largest positive eigenvalue of A_ξ. If no such μ exists, we simply insist that

$$1 < \coth r < \lambda .$$

By Proposition 2, we know that for $p = F(x, r\xi)$, L_p has a non-degenerate critical point at x. Also by Proposition 2, the index of L_p at x is equal to the multiplicity, say j, of the eigenvalue λ. If L_p is a Morse function, then the hypothesis of Theorem 2 imply that $j = n$, since we know $j > 0$. If L_p is not a Morse function, we know by Proposition 3 that there is a point $q \in H^{n+p}$, such that L_q is a Morse function having a critical point of index j. Again we conclude $j = n$. Thus λ is an eigenvalue of multiplicity n, and so $A_\xi = \lambda I$. q.e.d.

We remark that unlike the case for submanifolds of R^m, we cannot conclude immediately that f is an umbilical immersion because of the needed requirement in Proposition 4 that A_ξ must have an eigenvalue whose absolute value is greater than 1. Thus, further reasoning is necessary; the following proposition extends Proposition 4 to a local neighborhood U of x. This proposition is the key to overcoming the above-mentioned difficulties.

PROPOSITION 5. *Let $x \in M^n$ and suppose there is a unit length vector $\sigma \in T_x^\perp(M^n)$, such that A_σ has an eigenvalue whose absolute value is greater than* 1. *Then there is a neighborhood U of x in M^n such that f is umbilical on U and such that the second fundamental form $\alpha(X, Y)$ does not vanish on U.*

PROOF. Let V be a co-ordinate neighborhood of x and let $\xi_1, \cdots, \xi_p$ be orthonormal normal vector fields on V such that $\xi_1(x) = \pm\sigma$; the sign is chosen so that $A_{\xi_1(x)}$ has an eigenvalue $\beta > 1$.

Since the eigenvalues of A_{ξ_1} are continuous, there is a neighborhood U of x, U is contained in V, such that for any $u \in U$, $A_{\xi_1(u)}$ has an eigenvalue which is greater than 1. Thus $\alpha(X, Y)$ does not vanish on U.

We fix an arbitrary point $u \in U$. By Proposition 4 we know $A_{\xi_1(u)} = cI$ for some $c > 1$. Hence if the codimension $p = 1$, the proof is complete.

Assume $p > 1$. For the fixed $u \in U$, we define a function λ on $T_u^\perp(M^n)$ as follows. For any $\xi \in T_u^\perp(M^n)$, $\lambda(\xi)$ is the largest eigenvalue of A_ξ. We know λ is a continuous function on $T_u^\perp(M^n)$. Thus there is a neighborhood N of $\xi_1(u)$ in $T_u^\perp(M^n)$ such that $\lambda(\xi) > 1$ if $\xi \in N$. By Proposition 4, $A_\xi = \lambda(\xi)I$ if $\xi \in N$. Since N is open, we know that for each j there is a unit length vector $\xi \in N$ such that

$$\xi = a\xi_1 + b\xi_j \quad \text{for some} \quad a, b > 0 \quad \text{such that} \quad a^2 + b^2 = 1 .$$

We know

$$A_\xi = \lambda(\xi)I \tag{10}$$

but we have

$$A_\xi = A_{a\xi_1 + b\xi_j} = aA_{\xi_1} + bA_{\xi_j} . \tag{11}$$

Now $A_{\xi_1} = \lambda(\xi_1)I$ and thus (10) and (11) give

$$A_{\xi_j} = \frac{[\lambda(\xi) - a\lambda(\xi_1)]}{b} I .$$

Thus all the eigenvalues of A_{ξ_j} are the same, and we are justified in writing

$$A_{\xi_j} = \lambda(\xi_j)I \qquad 1 \leqq j \leqq p .$$

Then if $c_j \in R$, $1 \leqq j \leqq p$, we have

$$A_{\Sigma c_j \xi_j} = \sum_{j=1}^{p} c_j A_{\xi_j} = \sum_{j=1}^{p} c_j(\lambda(\xi_j)I) = \sum_{j=1}^{p} (c_j\lambda(\xi_j))I .$$

Hence,

$$\lambda\Big(\sum_{j=1}^{p} c_j\xi_j\Big) = \sum_{j=1}^{p} c_j\lambda(\xi_j) ,$$

and λ is a linear function on $T_u^\perp(M^n)$.

We have shown that for each $u \in U$, there is a linear function $\lambda(\xi)$ on $T_u^\perp(M^n)$ such that $A_\xi = \lambda(\xi)I$ for any $\xi \in T_u^\perp(M^n)$. This means that f is umbilical on U, and the proof is complete. q.e.d.

The following remark can be proven by methods similar to those employed by Cartan ([1], p.231); the proof is essentially the proper use of Codazzi's equation and is omitted here.

REMARK 1. Let U be a neighborhood of M^n on which the second fundamental form $\alpha(X, Y)$ does not vanish, and such that f is umbilical on U. Then the mean curvature vector η has constant length on U.

The following proposition and Proposition 5 will show that f is an umbilical immersion on M^n.

PROPOSITION 6. *The mean curvature vector η has constant length $\|\eta\| > 1$ on M^n.*

PROOF. Let $p \in H^{n+p}$ such that L_p is a Morse function. Since M^n is compact, there exists $x \in M^n$ such that L_p has a non-degenerate maximum at x. Hence the index of L_p at x is equal to n.

From Proposition 2, we know there exists $r > 0$ and a unit-length normal $\xi \in T_x^\perp(M^n)$ such that $p = F(x, r\xi)$, and we know $A_\xi = cI$ where $c > 1$. Proposition 5 implies that there is a linear function λ on $T_x^\perp(M^n)$ such that $A_\sigma = \lambda(\sigma)I$ for any $\sigma \in T_x^\perp(M^n)$.

Let $\xi_1, \cdots, \xi_p$ be an orthonormal basis for $T_x^\perp(M^n)$ such that $\xi_1 = \xi$. Then

$$\eta(x) = \sum_{j=1}^{p} \frac{(\text{trace } A_{\xi_j})}{n}\xi_j = \sum_{j=1}^{p} \frac{n\lambda(\xi_j)}{n}\xi_j = \sum_{j=1}^{p} \lambda(\xi_j)\xi_j ,$$

and so $A_{\eta(x)} = (\sum_{j=1}^{p} \lambda^2(\xi_j))I$.

Since $A_{\eta(x)} = g(\eta(x), \eta(x))I$, we conclude that

$$\|\eta(x)\|^2 = \sum_{j=1}^{p} \lambda^2(\xi_j) \geqq \lambda^2(\xi_1) > 1 .$$

Let $\beta = \|\eta(x)\|$ and let

$$S = \{u \in M^n \mid \|\eta(u)\| = \beta\} .$$

Since $\|\eta\|$ is continuous on M^n, we know S is closed. However, Proposition 5 and Remark 1 imply that S is open. Since $x \in S$, $S \neq \varnothing$, and the connectedness of M^n implies $S = M^n$. Thus we have $\|\eta\| = \beta > 1$ on M^n. q.e.d.

Now Propositions 5 and 6 imply that f is an umbilical immersion of M^n. As we remarked earlier, a compact umbilical M^n immersed H^{n+p} must be a metric sphere S^n, and the proof of Theorem 2 is complete.

3. A remark on the non-compact case. In this section, we note that a result corresponding to that of Nomizu and Rodriguez for the non-compact case does not hold. That is, let M^n be a connected, complete Riemannian manifold isometrically immersed in H^{n+p}. Assume that every Morse function of the form L_p, $p \in H^{n+p}$, has index 0 or n at any of its critical points. Then we *cannot* conclude that M^n is an umbilical submanifold of H^{n+p}.

The reason why the method of Nomizu and Rodriguez cannot be applied is that there may not be any focal points on the geodesic $\gamma(x, \xi, r)$ for some $x \in M^n$ and ξ a unit length vector in $T_x^{\perp}(M^n)$. In fact, this occurs if $|k_i| < 1$ for every eigenvalue k_i of A_ξ. Without the existence of a focal point on $\gamma(x, \xi, r)$, we cannot use the Index Theorem to prove $A_\xi = \lambda I$.

We supply here a simple example of a non-umbilic, complete surface M^2 embedded in H^3 such that every Morse function of the form L_p has index 0 at any of its critical points.

As before, we represent H^3 as a hypersurface of R^4; then the surface M^2 is defined by the global parametrization $y(s, t)$ as follows. Consider λ, μ such that $0 < \lambda < 1$ and $\mu = (1 - \lambda^2)^{1/2}$, then

$$y(s, t) = \frac{1}{\mu}(\cosh(\mu t)\cosh s, \lambda \cosh s, \sinh(\mu t)\cosh s, \mu \sinh s) .$$

Geometrically, M^2 is a cylinder in H^3 over the curve

$$\gamma(t) = \frac{1}{\mu}(\cosh(\mu t), \lambda, \sinh(\mu t), 0)$$

which has constant curvature λ.

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